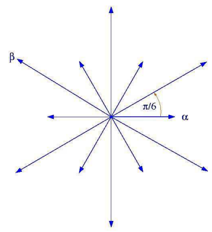
Etude de quelques sous-variétés des algèbres de Lie symétriques semi-simples.

Michaël Bulois



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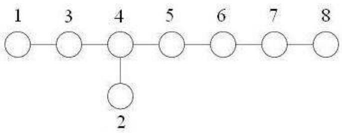

THÈSE

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Introduction

Ce travail traite de l'étude de certaines variétés des algèbres de Lie symétriques réductives, de dimension finie, définies sur un corps algébriquement clos de caractéristique nulle. Ces variétés sont des généralisations de sous-variétés déjà étudiées dans le cadre des algèbres de Lie.

Une algèbre de Lie symétrique est un couple $(\mathfrak{g}, \theta)$ où $\mathfrak{g}$ est une algèbre de Lie et θ une involution de $\mathfrak{g}$. Nous travaillerons uniquement dans le cas où $\mathfrak{g}$ est réductive de dimension finie sur $\Bbbk$, corps algébriquement clos de caractéristique 0. Notant $\mathfrak{k}$, respectivement $\mathfrak{p}$, le sous-espace propre associé à la valeur propre $+1$, resp. -1, de θ, on dispose de la décomposition d'espaces vectoriels suivante, dite décomposition de Cartan :

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

L'algèbre de Lie $\mathfrak{k}$ est alors réductive et $\mathfrak{p}$ est un $\mathrm{ad}(\mathfrak{k})$-module où $\mathrm{ad}(x)(\cdot) = [x, \cdot\,]$ est l'action adjointe. L'algèbre de Lie symétrique $(\mathfrak{g}, \theta)$ pourra alors aussi être notée $(\mathfrak{g}, \mathfrak{k})$ ou $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$.

Un cas particulier est celui où $\mathfrak{g}$ est isomorphe à la somme directe de deux copies $\mathfrak{g}_1$ et $\mathfrak{g}_2$ d'une algèbre de Lie $\mathfrak{g}'$, et où $\theta(\mathfrak{g}_i) = \mathfrak{g}_{3-i}$, $i = 1, 2$. Ce cas est dit *de type 0*, l'algèbre de Lie $\mathfrak{k} = \{x + \theta(x) \mid x \in \mathfrak{g}_1\}$ est alors isomorphe à $\mathfrak{g}'$ et le $\mathfrak{k}$-module $\mathfrak{p} = \{x - \theta(x) \mid x \in \mathfrak{g}_1\}$ est isomorphe au $\mathfrak{g}'$-module $\mathfrak{g}'$. Dans ce sens, les algèbres de Lie symétriques constituent une généralisation des algèbres de Lie.

Cette notion d'algèbre de Lie symétrique provient du cas où $\Bbbk$ est le corps des nombre complexes $\mathbb{C}$. Dans ce cas, considérons une forme réelle $\mathfrak{g}_{\mathbb{R}}$ de l'algèbre de Lie semi-simple complexe $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$, *i.e.* $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \oplus \imath\mathfrak{g}_{\mathbb{R}}$ avec $\imath^2 = -1$. Décomposons $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ où $\mathfrak{k}_{\mathbb{R}}$ (resp. $\mathfrak{p}_{\mathbb{R}}$) est la partie compacte (resp. non-compacte) de $\mathfrak{g}_{\mathbb{R}}$. Complexifiant cette égalité, on obtient $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ et $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}})$ est une algèbre de Lie symétrique. Inversement, toute algèbre de Lie symétrique complexe $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}})$ peut être obtenue par cette construction et ce, à partir d'une unique forme réelle à isomorphisme près. La correspondance entre les algèbres de Lie symétriques complexes et les algèbres de Lie réelles est approfondie par celle de Kostant-Sekiguchi [Se2]. Cette dernière met en bijection l'ensemble des orbites nilpotentes réelles de $\mathfrak{g}_{\mathbb{R}}$ et l'ensemble des orbites nilpotentes complexes de $\mathfrak{p}_{\mathbb{C}}$.

Dans le cas général, l'action de G, le *groupe algébrique adjoint* de l'algèbre de Lie $\mathfrak{g}$, donne une structure de G-variété algébrique à $\mathfrak{g}$. L'élément $g \in G$ opère sur $x \in \mathfrak{g}$ par $g.x = \mathrm{Ad}(g)(x)$.

Il s'avère alors naturel d'étudier certaines sous-G-variétés de $\mathfrak{g}$, ou de $\mathfrak{g}\times\mathfrak{g}$, dans le cadre de la *géométrie algébrique*. Dans un deuxième temps, on peut tenter de décrire dans le cas symétrique des variétés analogues aux précédentes. Ces variétés analogues sont des sous-K-variétés de $\mathfrak{p}$, ou de $\mathfrak{p}\times\mathfrak{p}$; ici $K\subset G$ est le sous-groupe algébrique connexe associé à $\mathfrak{k}$, il agit sur $\mathfrak{p}$ via l'action adjointe. Deux familles de variétés sont étudiées ici, les variétés commutantes et les nappes. Chacune donne lieu à un chapitre de ce manuscrit.

Les variétés commutantes

La première variété à laquelle nous nous intéressons est la variété commutante

$$\mathfrak{C}(\mathfrak{g}) := \{(x,y)\in\mathfrak{g}\times\mathfrak{g} \mid [x,y]=0\}\subset\mathfrak{g}\times\mathfrak{g}.$$

$\mathfrak{C}(\mathfrak{g})$ est une G-variété pour l'action diagonale de G sur $\mathfrak{g}\times\mathfrak{g}$ (*i.e.* $g.(x,y):=(g.x,g.y)$ pour tout $g\in G$). Les premiers travaux concernant $\mathfrak{C}(\mathfrak{g})$ ont été axés sur la question de son irréductibilité. Cette irréductibilité a tout d'abord été établie par Gerstenhaber [Ge] en type A, c'est à dire lorsque $\mathfrak{g}=\mathfrak{gl}_N$ est l'algèbre des matrices carrées de taille N (cf. [Gu, MT] pour une preuve plus concise). Richardson [Ri1] a ensuite généralisé ce résultat à toute algèbre de Lie réductive. Plus récemment, Popov [Po] a donné des résultats sur la dimension du lieu des singularités de $\mathfrak{C}(\mathfrak{g})$. Il existe toujours des questions ouvertes sur $\mathfrak{C}(\mathfrak{g})$, la plus notable étant de savoir si son idéal de définition est engendré par les polynômes de degré deux évidents.

Par ailleurs, diverses variétés reliées à $\mathfrak{C}(\mathfrak{gl}_N)$ ont été étudiées. Ces variétés peuvent être définies par une restriction sur l'hypothèse de commutativité. Par exemple, dans [Zo] on s'intéresse à la variété des matrices presque commutantes, formée par les couples de matrices (A,B) dont le commutateur $[A,B]$ est de rang 1, tandis que dans [Kn] il s'agit d'étudier le schéma commutateur diagonal, défini par les couples ayant un commutateur qui soit une matrice diagonale. Une autre famille de variétés reliées à $\mathfrak{C}(\mathfrak{gl}_N)$ est obtenue en imposant des restrictions sur l'ensemble des éléments commutants considérées. Ainsi, on peut trouver dans [Bar, Bas1, Bas2, BI, Bre] des études sur l'ensemble des paires commutantes de matrices symétriques, antisymétriques, nilpotentes ou de rang donné. Les différents travaux cités peuvent traiter de la description des composantes irréductibles ou bien de propriétés plus poussées de géométrie algébrique (normalité, intersection complète, Cohen-Macaulay, ...).

Une des variétés citées ci-dessus se généralise à toute algèbre de Lie réductive : la variété commutante nilpotente

$$\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}) := \{(x,y)\in\mathcal{N}(\mathfrak{g})\times\mathcal{N}(\mathfrak{g}) \mid [x,y]=0\} = \mathfrak{C}(\mathfrak{g})\cap(\mathcal{N}(\mathfrak{g})\times\mathcal{N}(\mathfrak{g})),$$

où $\mathcal{N}(\mathfrak{g})$ désigne l'ensemble des éléments nilpotents de $\mathfrak{g}$. Baranovsky [Bar] a montré que les composantes irréductibles de dimension maximale de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g})$ sont les ensembles de la forme

$$\mathfrak{C}(e) := \overline{G.(e,\mathcal{N}(\mathfrak{g})\cap\mathfrak{g}^e)}\subset\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}),\qquad \text{où } \mathfrak{g}^e:=\{x\in\mathfrak{g}\mid[x,e]=0\};$$

pour $e \in \mathfrak{g}$ nilpotent *distingué* (*i.e.* $\mathfrak{g}^e \subset \mathcal{N}(\mathfrak{g})$). Il conjecture alors que $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g})$ est équidimensionnelle et le prouve lorsque $\mathfrak{g}$ est de type A, à l'aide du schéma de Hilbert ponctuel.

Premet démontre dans [Pr] que $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g})$ est équidimensionnelle dans le cas général. Une partie de la preuve de Premet peut se résumer comme suit. Tout d'abord, il prouve qu'une composante irréductible de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g})$ est de la forme $\mathfrak{C}(e)$, où e satisfait $\mathfrak{g}^e \subset \overline{G.e}$. Un tel élément e est dit *self-large* et Premet montre que ces éléments font partie d'une classe plus vaste d'éléments dits *presque distingués*, *i.e.* dont le centralisateur dans $\mathfrak{g}$ est résoluble. La notion d'élément "self-large", que nous appellerons *quasi-distingué* faute d'une meilleure terminologie, est en réalité due à Panyushev (cf. [Pa5]) qui en donne une bonne caractérisation.

Concernant la généralisation aux algèbres de Lie symétriques, nous considérons les deux variétés

$$\mathfrak{C}(\mathfrak{p}) := \mathfrak{C}(\mathfrak{g}) \cap (\mathfrak{p} \times \mathfrak{p}) \text{ et } \mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}) = \mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}) \cap (\mathfrak{p} \times \mathfrak{p}).$$

La question de l'irréductibilité de $\mathfrak{C}(\mathfrak{p})$ a été étudiée par Panyushev, Panyushev et Yakimova, Sabourin et Yu, cf. [Pa1, Pa2, Pa3, PY, SY1, SY2]. Cette question est encore ouverte dans trois cas particuliers, voir [PY]. Dans les autres cas, il a été déterminé si $\mathfrak{C}(\mathfrak{p})$ est irréductible ou non.

Le premier chapitre de cette thèse est consacré a l'étude de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$. Ce travail s'articule autour de deux point principaux. Dans un premier temps, il a été possible de généraliser certaines notions de la preuve de Premet aux algèbres de Lie symétriques. En particulier, il s'est avéré que les notions d'éléments nilpotents $\mathfrak{p}$-*distingués*, *quasi*-$\mathfrak{p}$-*distingués* (ou encore "$\mathfrak{p}$-*self-large*") et *presque* $\mathfrak{p}$-*distingués* constituent une bonne généralisation des éléments distingués, quasi-distingués (i.e. "self-large") et presque distingués utilisés par Premet. Il a pu être démontré que les composantes irréductibles de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$ sont nécessairement de la forme

$$\mathfrak{C}(e) := \overline{K.(e, \mathcal{N}(\mathfrak{g}) \cap \mathfrak{p}^e)}$$

pour e quasi-$\mathfrak{p}$-distingué. De plus, comme pour des algèbres de Lie, un tel élément quasi-$\mathfrak{p}$-distingué est aussi presque $\mathfrak{p}$-distingué.

Dans un second temps, une énumération explicite des éléments presque $\mathfrak{p}$-distingués à été obtenue au cas par cas. Ceci donne en passant une classification similaire à celle des éléments compacts étudiés par Popov-Tevelev [PT]. De plus, à partir de cette classification, il a été possible d'obtenir une énumération complète des éléments quasi-$\mathfrak{p}$-distingués. Ces résultats, combinés à quelques autres méthodes géométriques, a permis d'aboutir, dans un certain nombre de cas (quinze sur vingt), à une vérification de la conjecture suivante :

Conjecture A. *Les composantes irréductibles de* $\mathfrak{C}^{nil}(\mathfrak{p})$ *sont les* $\mathfrak{C}(e)$ *pour* e *élément* $\mathfrak{p}$*-distingué.*

Ce premier travail a donné lieu à la publication [Bu1].

Les nappes (sheets)

Les nappes sont un second type de sous-variétés qui ont aussi fait l'objet de nombreuses recherches. Une nappe est une composante irréductible d'un ensemble de la forme

$$\mathfrak{g}^{(m)} := \{x \in \mathfrak{g} \mid \dim G.x = m\}; \quad m \in \mathbb{N}.$$

Ces nappes apparaissent dans plusieurs problèmes et la connaissance de leur structure permet de transférer certaines propriétés des éléments semi-simples aux éléments nilpotents. Par exemple, un point crucial de la preuve par Richardson [Ri1] de l'irreductibilité de $\mathfrak{C}(\mathfrak{g})$ s'interprète naturellement en terme de nappes.

Les nappes contenant des éléments semi-simples ont tout d'abord été introduites par Dixmier dans [Di1, Di2]. Kraft donna ensuite dans [Kr] une paramétrisation des nappes dans le cas $\mathfrak{g} = \mathfrak{gl}_N$. Cette paramétrisation a été ensuite généralisée par Bohro et Kraft dans [Boh, BK] pour $\mathfrak{g}$ quelconque. Les notions de *polarisation*, d'*induction d'orbites* et de *classes de Jordan* jouent un rôle central dans cette paramétrisation.

La *classe de Jordan*, plus souvent appelée *classe de décomposition*[1], d'un élément $x \in \mathfrak{g}$ est définie par

$$J_G(x) := \{y \in \mathfrak{g} \mid \exists g \in G, g.\mathfrak{g}^x = \mathfrak{g}^y\}.$$

Les classes de Jordan sont donc des classes d'équivalence et, dans le cas $\mathfrak{g} = \mathfrak{gl}_N$, deux matrices sont dans la même classe de Jordan si et seulement si elles ont des blocs de Jordan de même taille ayant éventuellement des valeurs propres différentes. On peut montrer que $\mathfrak{g}$ est l'union finie de classes de Jordan. Il est alors facile de voir que les nappes sont des unions finies de ces classes. Une grande partie du travail de [Boh, BK] consiste à caractériser les nappes par leurs classes de Jordan. En un sens, les classes de Jordan généralisent les orbites nilpotentes de par leurs propriétés (nombre fini, lissité, locale fermeture,...). Plus tard, [Bro] a étudié plus en détail la géométrie de $J_G(x)$, $\overline{J_G(x)}$ ou de la normalisation de $\overline{J_G(x)}$.

Il est important de noter que la notion de nappe inclut le cas particulier de l'ensemble des éléments réguliers de $\mathfrak{g}$. Rappelons que l'ensemble des éléments réguliers d'une G-variété Z est l'ensemble des éléments ayant une orbite de dimension maximale dans Z. Pour cette nappe régulière S^{reg}, un résultat de Kostant peut s'interpréter comme l'existence d'un quotient géométrique S^{reg}/G (cf. [Ko]). De plus, ce quotient est isomorphe à un espace affine. Rubenthaler a ensuite généralisé ceci à une classe de nappes (dites *admissibles*) dans [Ru]. Dans [Kat], Katsylo a prouvé l'existence d'un quotient géométrique pour une nappe quelconque. Plus récemment, Im Hof [IH] a obtenu la *lissité* des nappes dans les cas classiques.

Les articles [Ko, Ru, Kat, IH] utilisent une paramétrisation des nappes différente de celle introduite dans [Kr, Boh, BK] ; cette paramétrisation repose sur la construction suivante. Si

[1] Nous avons choisi d'utiliser ici l'appellation « classe de Jordan » afin de faciliter les références au livre de Tauvel et Yu, où elle est employée. Signalons à ce propos qu'afin de ne pas alourdir la présentation, nous renvoyons souvent à [TY] pour citer des résultats qui ont été obtenus par différents auteurs.

e est un élément nilpotent appartenant à une nappe S, on peut inclure e dans un $\mathfrak{sl}_2$-triplet (e,h,f), (*i.e.* $[h,e]=2e$, $[h,f]=-2f$ et $[e,f]=h$). On peut alors définir une *tranche de Slodowy* de S par

$$e+X := (e+\mathfrak{g}^f)\cap S.$$

L'étude de la tranche $e+\mathfrak{g}^f$ à d'abord été faite par Slodowy dans [Sl, § 7.4]. Katsylo montre que $S=G.(e+X)$ et que S/G s'identifie à un quotient de $e+X$ par un groupe fini. Par ailleurs, Im-Hof a prouvé que le morphisme $G\times(e+X)\to S$ est lisse.

Les K-*nappes* sont les composantes irréductibles d'ensembles de la forme

$$\mathfrak{p}^{(m)} := \{x\in\mathfrak{p}\mid \dim K.x=m\},\quad m\in\mathbb{N}.$$

Kostant et Rallis [KR] ont étudié le cas de la nappe régulière (*i.e.* lorsque m est maximal) et ont obtenu des résultats généralisant ceux de [Ko]. Par ailleurs, des K-nappes contenant des éléments semi-simples (ou K-nappes *de Dixmier*) ont déjà été utilisées dans [Pa3, SY2, PY] pour étudier l'irréductibilité de $\mathfrak{C}(\mathfrak{p})$ (cf. [SY2, Théorème 2.1]). Le principal résultat alors utilisé est que des éléments nilpotents particuliers (les éléments *pairs*) sont dans des K-nappes de Dixmier.

Le second chapitre de cette thèse porte sur une étude (en anglais) plus poussée des K-nappes. Des méthodes telles que l'induction d'orbites ne se généralisent pas de façon simple aux K-nappes. Par exemple, la notion d'induction présentée, dans [Oh3], ne préserve pas la dimension des orbites en général. Il est alors utile d'observer que

$$\mathfrak{p}^{(m)}\subset\mathfrak{g}^{(2m)}.$$

En fait, l'intersection de $\mathfrak{p}$ et d'une G-orbite adjointe $\mathcal{O}$ est soit vide, soit une réunion finie de K-orbites qui sont des sous-variétés lagrangiennes de $\mathcal{O}$ (pour la forme symplectique naturelle). En particulier, ces K-orbites sont de dimension m si $\dim\mathcal{O}=2m$. Cela a motivé l'étude de l'intersection d'une G-nappe S avec $\mathfrak{p}$ dans le but d'obtenir des informations sur les K-nappes.

Un analogue symétrique des classes de Jordan, les K-classes de Jordan, avait déjà été étudié par Tauvel-Yu dans [TY, §39]. La K-classe de Jordan d'un élément $x\in\mathfrak{p}$ peut être définie par

$$J_K(x) := \{y\in\mathfrak{p}\mid \exists k\in K,\ k.\mathfrak{p}^x=\mathfrak{p}^y\}.$$

Pour l'étude des K-nappes, il a fallu obtenir le résultat important suivant. Si J est une classe de Jordan intersectant $\mathfrak{p}$ alors la variété $J\cap\mathfrak{p}$ est lisse et équidimensionnelle. De plus ses composantes irréductibles sont les K-classes de Jordan contenues dans $J\cap\mathfrak{p}$.

Concernant les K-nappes, la *lissité* a été obtenue dans les cas classiques. Ensuite, si S est une nappe de $\mathfrak{g}$, quelques hypothèses ont été introduites afin d'obtenir une description de $S\cap\mathfrak{p}$ à l'aide de la tranche de Slodowy. Ainsi, sous ces hypothèses, il a été possible de montrer la formule suivante de *paramétrisation* pour une nappe S :

$$S\cap\mathfrak{p} = \bigcup_{K.e\subset G.e\cap\mathfrak{p}} \overline{K.(e+X\cap\mathfrak{p})}^{\bullet},$$

où $Z^\bullet$ désigne l'ensemble des éléments réguliers de Z. Ceci a en outre permis de prouver *l'équi-dimensionnalité* de $S \cap \mathfrak{p}$.

Les hypothèses évoquées ci-dessus ont été démontrées pour les algèbres de Lie symétriques $(\mathfrak{g}, \mathfrak{k})$ lorsque $\mathfrak{g} = \mathfrak{gl}_N$. Dans ces algèbres de Lie symétriques, on obtient alors une description complète des composantes irréductibles de $S \cap \mathfrak{p}$. On donne en particulier la dimension de $S \cap \mathfrak{p}$. Ceci a notamment permis d'aboutir à une description intéressante des K-nappes. De plus, peu de travail supplémentaire a alors suffi pour caractériser les K-orbites nilpotentes formant une nappe (orbites rigides), ou les K-nappes contenant des éléments semi-simples (K-nappes de Dixmier).

Ce travail a fait l'objet d'une prépublication en ligne (cf. [Bu2]).

Remerciements.

Je tiens tout d'abord à remercier Thierry Levasseur pour sa disponibilité, sa persévérance dans les lectures de premières versions obscures et sa rigueur mathématique. Bien évidemment, je remercie aussi Michel Brion et Dmitri Panyushev pour avoir accepté le travail de rapporteur de cette thèse. Je leur suis reconnaissant pour leur lecture attentive d'une première version de ce manuscrit qu'ils ont aidé à améliorer par leurs remarques très pertinentes. Je remercie également Rupert Yu et Johannes Huisman qui ont accepté de faire partie du jury. Merci aussi aux personnes qui se sont intéressées à mon travail, en particulier les membres du laboratoire de Poitiers. Enfin, je remercie tous ceux qui m'ont donné envie de recherche en mathématiques, à commencer par mes directeurs de stages successifs Abbas Chazad Movahhedi, Bertrand Rémy et Philippe Caldero.

Par ailleurs, je tiens à rappeler ici, et beaucoup de personnes peuvent en témoigner, qu'une thèse peut facilement devenir soit un outil d'exploitation de jeunes chercheurs soit une activité d'encadrement méprisée. Cette thèse ne relevant clairement pas de ces définitions, je remercie donc toutes les personnes de bonne volonté qui ont contribué à créer un cadre propice pour accomplir ce travail, en particulier les membres du laboratoire.

D'un point de vue bibliographique, je me dois de citer [TY] qui, d'une part, m'a appris toute la théorie et, d'autre part, à répondu à nombre de mes questions. Merci aussi à [Wi] d'avoir répondu à d'innombrables autres questions. Par ailleurs, les photographies de la page de garde sont extraites de [Haw].

Finalement, je voudrais remercier aussi mon entourage. Mes parents sans qui je ne serais pas là ; Nolwenn, avec deux n ; notre chat Lumo et tous mes amis. Ils sont cools.

Chapitre I

Variété commutante nilpotente

Introduction

Soit $\mathfrak{g}$ une algèbre de Lie réductive, de dimension finie, définie sur un corps $\Bbbk$ algébriquement clos de caractéristique zéro. Soit G son groupe adjoint de sorte que $[\mathfrak{g}, \mathfrak{g}] = \mathrm{Lie}(G)$. Soit θ un automorphisme involutif de $\mathfrak{g}$ et $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ la décomposition associée en θ-espaces propres de valeurs propres respectives $+1$ et -1. Ceci nous donne une algèbre de Lie symétrique $(\mathfrak{g}, \mathfrak{k})$. Notons K le sous-groupe algébrique connexe de G qui a pour algèbre de Lie $\mathfrak{k} \cap [\mathfrak{g}, \mathfrak{g}]$. Le travail de R.W. Richardson [Ri1] a montré que $\mathfrak{C}(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\}$, la variété commutante de $\mathfrak{g}$, est irréductible. Suivant une conjecture [Bar] de V. Baranovsky, A. Premet a montré [Pr] que $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g})$, la variété commutante nilpotente de $\mathfrak{g}$ est équidimensionnelle et a indexé ses composantes irréductibles par les orbites distinguées. Par ailleurs, l'irréductibilité de la variété commutante de $\mathfrak{p}$, $\mathfrak{C}(\mathfrak{p})$, a été étudiée dans [PY, Pa1, Pa2, Pa3, SY1, SY2]. Nous nous intéressons à $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$, la variété commutante nilpotente de $\mathfrak{p}$. Soit $\mathcal{N}$ le cône des éléments nilpotents de $\mathfrak{p}$, on note

$$\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}) = \{(x, y) \in \mathcal{N} \times \mathcal{N} : [x, y] = 0\} = \mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}) \cap (\mathfrak{p} \times \mathfrak{p})$$

la variété commutante nilpotente de $\mathfrak{p}$. Le but de ce travail est d'établir, dans beaucoup de cas, la conjecture suivante. (Rappelons qu'un élément de $\mathfrak{p}$ est dit $\mathfrak{p}$-distingué si son centralisateur dans $\mathfrak{p}$ ne contient que des éléments nilpotents.)

Conjecture A. *La variété $\mathfrak{C}^{nil}(\mathfrak{p})$ est équidimensionnelle de dimension* $\dim \mathfrak{p}$. *Ses composantes irréductibles sont indexées par les orbites d'éléments $\mathfrak{p}$-distingués.*

Il est facile de voir qu'il suffit de prouver le résultat lorsque la paire symétrique $(\mathfrak{g}, \mathfrak{k})$ est irréductible. En adoptant les notations de [He1, p. 518], nous établissons la conjecture dans les cas AIII, CII, DIII, E{II–IX}, FI, FII et GI. Dans les autres cas nous obtenons des résultats qui renforcent sa validité.

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Key words and phrases : semisimple Lie algebra, symmetric pair, commuting variety, nilpotent orbit.

La nature des composantes irréductibles potentielles de $\mathfrak{p}$ peut être comprise à l'aide de considérations générales contenues dans la première section. Certaines des méthodes utilisées généralisent celles de [Pr] ; on est en particulier amené à introduire la notion d'élément presque $\mathfrak{p}$-distingué (cf. 1.4.1). Un tel élément e, s'il n'est pas $\mathfrak{p}$-distingué, définit une variété $\mathfrak{C}(e)$ susceptible d'être une composante, dite étrange, de dimension $< \dim \mathfrak{p}$ dans $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$. L'essentiel du travail consiste donc à montrer que les variétés de la forme $\mathfrak{C}(e)$ pour de tels e ne fournissent pas de composantes irréductibles.

Les sections 2 et 3 donnent une classification des éléments presque $\mathfrak{p}$-distingués en termes d'(ab-)diagrammes de Young (cf. [Oh1, Oh2]) dans le cas où $\mathfrak{g}$ est classique. Cette classification permet de prouver la conjecture dans les cas AIII, CII et DIII.

Dans les sections 4 et 5 on montre qu'un certain nombre de composantes étranges ne peuvent apparaître dans les cas AI, AII, CI et BDI. Ces résultats assurent, par exemple, que la conjecture est vraie dans les cas AI en rang $\leqslant 4$, AII en rang $\leqslant 3$, CI en rang $\leqslant 7$ et BDI en rang $\leqslant 2$.

La section 6 traite du cas où $\mathfrak{g}$ est exceptionnelle. À l'aide de tables établies par D. Z. Djokovic on y démontre la conjecture dans tous les cas, sauf celui de EI où deux composantes étranges restent à éliminer.

L'appendice 7 est un complément permettant de décrire une classe d'éléments appelés quasi-$\mathfrak{p}$-distingués. Ces éléments sont intimement liés à la méthode issue de la section 1.4 visant à éliminer un certain nombre de composantes étranges. Il fait suite à l'article de D. Panyushev [Pa5] qui, dans le cas des algèbres de Lie, a défini la notion d'élément "self-large" que nous appelons ici quasi-distingué.

Remerciements. D. Panyushev nous a indiqué qu'il a obtenu des résultats semblables aux nôtres. Nous le remercions de nous en avoir informé. Je tiens également à remercier le rapporteur pour ses très pertinentes remarques et suggestions.

1 Généralités

Rappelons quelques résultats tirés de [KR]. Tout élément $t \in \mathfrak{p}$ s'écrit de façon unique $t = s + n$ où s et n sont des éléments de $\mathfrak{p}$ respectivement semi-simple et nilpotent (via l'action adjointe $\mathrm{ad}_{\mathfrak{g}}$ sur $\mathfrak{g}$). On appelle tore de dimension r toute algèbre de Lie commutative constituée d'éléments semi-simples ; on notera souvent T_r un tel tore. On appelle rang de $\mathfrak{p}$ et on note $\mathrm{rk}(\mathfrak{p})$ la dimension commune des tores maximaux de $\mathfrak{p}$. L'ensemble $\mathcal{N}$ est un cône ; qui est une variété équidimensionnelle de dimension $\dim \mathfrak{p} - \mathrm{rk}(\mathfrak{p})$. Le cône $\mathcal{N}$ est stable sous l'action de K et se décompose en un nombre fini d'orbites. On notera $\mathcal{O}(e)$ la K-orbite d'un élément nilpotent e. Si $X \subset \mathfrak{g}$ est une partie quelconque de $\mathfrak{g}$, on note $\mathcal{N}(X)$ l'ensemble des éléments nilpotents de $\mathfrak{g}$ contenus dans X. Pour $x \in \mathfrak{g}$ on pose $X^x = \{y \in X \mid [x, y] = 0\}$. Rappelons aussi que pour tout élément $e \in \mathfrak{p}$, on a [KR, Proposition 5]

$$\dim \mathcal{O}(e) = \dim \mathfrak{k} - \dim \mathfrak{k}^e = \dim \mathfrak{p} - \dim \mathfrak{p}^e.$$

Enfin lorsqu'il n'y aura pas d'ambiguïté, pr_1 désignera une application de projection sur la première variable.

1.1 Réduction au cas simple

On pose $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$; $\mathfrak{k}' = \mathfrak{g}' \cap \mathfrak{k}$ et $\mathfrak{p}' = \mathfrak{g}' \cap \mathfrak{p}$. On a alors $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$, ce qui nous donne une paire symétrique semi-simple $(\mathfrak{g}', \mathfrak{k}')$. Notons que d'après la définition de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$, on a $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}) = \mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}')$. Pour l'étude de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$, on peut donc supposer sans perte de généralité que $\mathfrak{g}$ est semi-simple. Ce sera le cas dans tout le reste de l'article.

De plus, si $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ est une décomposition de $\mathfrak{g}$ en algèbres de Lie simples (cf. [TY, 20.1.7]), l'action de θ envoie chaque $\mathfrak{g}_i$ sur un $\mathfrak{g}_{\theta(i)}$. On a alors deux cas possibles :

a) $i = \theta(i)$, dans ce cas $\mathfrak{g}_i$ est θ-stable et $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$

b) $i \neq \theta(i)$; dans ce cas on peut supposer que $\mathfrak{g}_i \times \mathfrak{g}_{\theta(i)} \cong \mathfrak{g}_i \times \mathfrak{g}_i$ avec $\theta(x, y) = (y, x)$. On pose $\mathfrak{k}_i = \{(x, x) \mid x \in \mathfrak{g}_i\}$ et $\mathfrak{p}_i = \{(x, -x) \mid x \in \mathfrak{g}_i\}$.

On a alors $\mathcal{N}(\mathfrak{p}) = \bigoplus_i \mathcal{N}(\mathfrak{p}_i)$ et $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}) = \bigoplus_i \mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}_i)$.

Par ailleurs, dans le cas b), on a $\mathfrak{p}_i \cong \mathfrak{g}_i$ par $(x, -x) \mapsto x$. Cet isomorphisme envoie $\mathcal{N}(\mathfrak{p}_i)$ sur $\mathcal{N}(\mathfrak{g}_i)$ et $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}_i)$ sur $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}_i)$, dont Premet a décrit les composantes irréductibles [Pr].

La classification des composantes irréductibles de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}_i)$ pour $\mathfrak{g}_i$ simple, suffit donc pour obtenir la classification des composantes irréductibles de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$.

1.2 Paramétrisation par les orbites

Rappelons que $\mathcal{N}$ désigne le cône des éléments nilpotents de $\mathfrak{p}$. Soit $e \in \mathcal{N}$. Pour $e = 0$ on pose $\mathfrak{g}(e, 0) = \mathfrak{g}$ et $\mathfrak{g}(e, i) = 0$ si $i \in \mathbb{Z}^*$. Si $e \neq 0$, il existe un $\mathfrak{sl}_2$-triplet normal (e, h, f) contenant e, c'est à dire : (e, h, f) est un $\mathfrak{sl}_2$-triplet avec $e, f \in \mathfrak{p}$ et $h \in \mathfrak{k}$ (cf. [KR, Proposition 4]). On pose $\mathfrak{g}(i, h) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$ et on a $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i, h)$ (cf. [TY, 19.2.7]). Comme $h \in \mathfrak{k}$, cette décomposition est θ-stable et on pose $\mathfrak{k}(i, h) = \mathfrak{g}(i, h) \cap \mathfrak{k}$ et $\mathfrak{p}(i, h) = \mathfrak{g}(i, h) \cap \mathfrak{p}$. On pose, par ailleurs, $\mathfrak{g}(e, i) = \mathfrak{g}(i, h) \cap \mathfrak{g}^e$. Comme $e \in \mathfrak{p}$, $\mathfrak{g}(e, i)$ est θ-stable et on note $\mathfrak{p}(e, i) = \mathfrak{g}(e, i) \cap \mathfrak{p}$. On a alors

$$\mathfrak{p}^e = \bigoplus_{i \in \mathbb{N}} \mathfrak{p}(e, i).$$

On sait que $\mathfrak{g}(e, 0)$ est le stablisateur de (e, h, f) dans $\mathfrak{g}$, c'est donc une sous-algèbre réductive dans $\mathfrak{g}$ (cf. [TY, 20.5.13]). L'ensemble de ses éléments nilpotents est donc $\mathcal{N}(\mathfrak{g}) \cap \mathfrak{g}(e, 0)$ et on a

$$\mathcal{N}(\mathfrak{g}) \cap \mathfrak{g}^e = \mathcal{N}(\mathfrak{g}(e, 0)) \times \bigoplus_{i > 0} \mathfrak{g}(e, i).$$

Par la discussion ci-dessus sur la θ-stabilité, on a de plus que :

$$\mathcal{N} \cap \mathfrak{p}^e = \mathcal{N}(\mathfrak{p}(e, 0)) \times \bigoplus_{i > 0} \mathfrak{p}(e, i).$$

C'est une variété qui contient le même nombre de composantes irréductibles que $\mathcal{N}(\mathfrak{p}(e,0))$ que l'on indexe par un ensemble I_e. On a alors :

$$\mathcal{N} \cap \mathfrak{p}^e = \bigcup_{j \in I_e} \mathcal{N}_e^{(j)} \qquad \text{où} \qquad \mathcal{N}_e^{(j)} = (\mathcal{N}(\mathfrak{p}(e,0))_j \times \bigoplus_{i>0} \mathfrak{p}(e,i).$$

Définition 1.2.1. On pose

$$\mathfrak{C}(e)^{(j)} := \overline{\operatorname{Ad} K.(e, \mathcal{N}_e^{(j)})} \subset \mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}) \quad \text{et} \quad \mathfrak{C}(e) = \overline{\operatorname{Ad} K.(e, \mathcal{N} \cap \mathfrak{p}^e)} = \bigcup_{j \in I_e} \mathfrak{C}(e)^{(j)}.$$

On dit que e engendre $\mathfrak{C}(e)^{(j)}$.

Les sous-variétés du type $\mathfrak{C}(e)^{(j)}$ sont irréductibles. Or par [KR, Théorème 2], il existe un nombre fini de K-orbites nilpotentes dans $\mathfrak{p}$, dont on notera $e_1, \dots, e_k$ des représentants, de sorte que :

$$\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}) = \bigcup_{\substack{i=1,\dots,k \\ j \in I_{e_i}}} \mathfrak{C}(e_i)^{(j)}.$$

Cette union étant finie, on peut en déduire que les composantes irréductibles de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$ sont de la forme $\mathfrak{C}(e_i)^{(j)}$ pour $i \in [\![1;k]\!]$ et $j \in I_{e_i}$.

1.3 Éléments $\mathfrak{p}$-distingués et conjecture

La notion d'élément nilpotent $\mathfrak{p}$-distingué va s'avérer très importante pour la suite.

Définition 1.3.1. Soit $e \in \mathcal{N}$. L'élément e est dit $\mathfrak{p}$-distingué si $\mathfrak{p}^e \subset \mathcal{N}$.

Lemme 1.3.2. *L'élément e est $\mathfrak{p}$-distingué si et seulement si $\mathfrak{p}(e,0) = \{0\}$, ce qui revient à dire que $\mathfrak{p}^e = \bigoplus_{i>0} \mathfrak{p}(e,i)$.*

Démonstration. Si $\mathfrak{p}(e,0) = \{0\}$ alors $\mathfrak{p}^e = \bigoplus_{i>0} \mathfrak{p}(e,i) \subset \mathcal{N}$. Réciproquement, si $\mathfrak{p}(e,0) \neq \{0\}$ alors $\mathfrak{p}(e,0)$ est le -1-espace propre de $\mathfrak{g}(e,0)$ qui est réductif dans $\mathfrak{g}$. Il contient donc des éléments semi-simples non-triviaux et e n'est pas distingué. $\square$

Soit $e \in \mathcal{N}$. On veut calculer la dimension des $\mathfrak{C}(e)^{(j)}$. On s'intéresse pour cela à l'application dominante entre variétés irréductibles

$$\xi : \left\{ \begin{array}{ccc} K \times \mathcal{N}_e^{(j)} & \to & \mathfrak{C}(e)^{(j)} \\ (g,x) & \mapsto & (g.e, g.x) \end{array} \right. .$$

Pour tout $x \in \mathcal{N}_e^{(j)}$, la fibre $\xi^{-1}(\xi(1,x))$ est l'ensemble des $(g, g^{-1}.x)$ avec $g \in K^e$. On en déduit que $\dim \xi^{-1}(\xi(1,x)) = \dim K^e = \dim \mathfrak{k}^e$. Ceci reste vrai pour toute fibre non-vide, et en notant $p = \dim \mathfrak{p}$ on a (cf. [TY, Théorème 15.5.3])

$$\begin{aligned} \dim \mathfrak{C}(e)^{(j)} &= \dim K + \dim \mathcal{N}_e^{(j)} - \dim \mathfrak{k}^e \\ &= p - (\dim \mathfrak{p}^e - \dim \mathcal{N}_e^{(j)}) \\ &= p - \operatorname{codim}_{\mathfrak{p}(e,0)} \mathcal{N}_e^{(j)} \\ &= p - \operatorname{rk}(\mathfrak{p}(e,0)) \end{aligned}$$

où la dernière égalité découle de [KR, Théorème 3]. On remarque que les sous-variétés $\mathfrak{C}(e)$ sont équidimensionnelles.

Définition 1.3.3. Pour un élément nilpotent $e \in \mathfrak{p}$, on définit le défaut de e comme étant le rang de $\mathfrak{p}(e,0)$. On le note $\delta(e)$. Le défaut est invariant sous l'action de K.

Proposition 1.3.1. *La variété $\mathfrak{C}^{nil}(\mathfrak{p})$ est de dimension p et ses composantes irréductibles de dimension maximale sont les $\mathfrak{C}(e)$ avec e représentant d'orbite $\mathfrak{p}$-distinguée.*

Démonstration. Par le calcul précédent, on voit que $\dim \mathfrak{C}(e)^{(j)} = p$ si et seulement si e est de défaut nul, ce qui est équivalent à dire que e est $\mathfrak{p}$-distingué. Dans le cas contraire, on a $\dim \mathfrak{C}(e)^{(j)} < p$. Par ailleurs, comme les éléments nilpotents $\mathfrak{p}$-réguliers sont $\mathfrak{p}$-distingués, il existe des éléments $\mathfrak{p}$-distingués ce qui suffit à montrer que $\dim \mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}) = p$. Maintenant, si e est $\mathfrak{p}$-distingué, $\mathcal{N} \cap \mathfrak{p}^e = \mathfrak{p}^e$ donc $\mathfrak{C}(e)$ est irréductible. On a de plus $\mathrm{pr}_1(\mathfrak{C}(e)) = \overline{K.e}$. On en déduit que les $\mathfrak{C}(e_i)$ sont distincts pour des éléments e_i appartenant à des K-orbites différentes. □

On appellera *composante étrange* une composante irréductible de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$ de dimension strictement inférieure à p.

Remarque 1.3.4. En rang 0, *i.e.* si $\mathfrak{p} = \{0\}$ ou encore si θ est triviale, on a $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p}) = \{0\} \times \{0\} = \mathfrak{C}(0)$.

En rang 1, le commutant d'un élément $x \in \mathfrak{p}$ est $\Bbbk x$. Tous les éléments nilpotents sont donc $\mathfrak{p}$-distingués et les composantes irréductibles de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$ sont les $\mathfrak{C}(e)$ où e est nilpotent.

1.4 Éléments presque $\mathfrak{p}$-distingués

Nous allons maintenant identifier les composantes étranges en introduisant la notion d'élément presque $\mathfrak{p}$-distingué, qui généralise celle d'élément presque distingué (cf. [Pr]). Les résultats qui suivent, et particulièrement la proposition 1.4.1, sont largement inspirés de [Pr, Proposition 2.1].

Définition 1.4.1. On dit que $e \in \mathcal{N}$ est presque $\mathfrak{p}$-distingué si $\mathfrak{p}(e,0)$ ne contient pas d'élément nilpotent non nul, c'est à dire si $\mathfrak{p}(e,0)$ est un tore $T_{\delta(e)}$. C'est équivalent au fait que $\delta(e) = \dim \mathfrak{p}(e,0)$.

Lemme 1.4.2. *Si $e \in \mathcal{N}$ et $j \in I_e$ sont tels que $\mathfrak{C}(e)^{(j)}$ est une composante irréductible de $\mathfrak{C}^{nil}(\mathfrak{p})$, alors $\mathcal{N}_e^{(j)} \subseteq \overline{\mathcal{O}(e)}$.*

Démonstration. Remarquons tout d'abord que le groupe $\mathrm{GL}(2)$ agit sur $\mathfrak{p} \times \mathfrak{p}$ via :

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} .(x,y) = (\alpha x + \beta y, \gamma x + \delta y).$$

Comme toute combinaison linéaire d'éléments commutant dans $\mathcal{N}$ est encore dans $\mathcal{N}$, la variété $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$ est $\mathrm{GL}(2)$-invariante. Puisque $\mathrm{GL}(2)$ est connexe, il stabilise la composante irréductible

$\mathfrak{C}(e)^{(j)}$. En particulier $\mathfrak{C}(e)^{(j)}$ est stable sous l'application $\sigma : (x,y) \to (y,x)$ de $\mathfrak{p} \times \mathfrak{p}$. Comme $\mathrm{pr}_1(K.(e, \mathcal{N}_e^{(j)})) = \mathcal{O}(e)$, on a $\mathrm{pr}_1(\mathfrak{C}(e)^{(j)}) \subseteq \overline{\mathcal{O}(e)}$. Ceci implique en particulier que

$$\mathcal{N}_e^{(j)} = (\mathrm{pr}_1 \circ \sigma)(e, \mathcal{N}_e^{(j)}) \subseteq \overline{\mathcal{O}(e)}.$$

□

Notons ici une remarque liée à la preuve précédente : si $\mathcal{N}_e^{(j)} \subseteq \overline{\mathcal{O}(e)}$ alors $\mathrm{GL}(2).\mathfrak{C}(e)^{(j)} = \mathfrak{C}(e)^{(j)}$. On ne peut donc pas obtenir plus de résultats en considérant l'action de $\mathrm{GL}(2)$ à la place de celle de σ.

Lemme 1.4.3. *On a $\bigoplus_{i \geqslant 2} \mathfrak{p}(e,i) \subset \overline{\mathcal{O}(e)}$.*

Démonstration. On peut supposer $e \neq 0$. On considère la sous-algèbre parabolique $\mathfrak{q} = \bigoplus_{i \geqslant 0} \mathfrak{k}(i,h)$ de l'algèbre de Lie $\mathfrak{k}$. D'après [TY, 29.4.3], il existe un sous-groupe algébrique Q de K ayant $\mathfrak{q}$ pour algèbre de Lie. On a alors $[\mathfrak{q}, e] = \bigoplus_{i \geqslant 2} \mathfrak{p}(i,h)$, donc $Q.e$ est une sous-variété de $\bigoplus_{i \geqslant 2} \mathfrak{p}(i,h)$ de même dimension. On en déduit que $\overline{Q.e} = [\mathfrak{q}, e] = \bigoplus_{i \geqslant 2} \mathfrak{p}(i,h)$. L'affirmation du lemme est alors immédiate. □

Pour éliminer des possibilités de composantes étranges, nous allons utiliser le lemme 1.4.2. D'après le lemme 1.4.3, cela ne peut éventuellement s'appliquer que dans le cas où $\mathfrak{p}(e,0)$ ou $\mathfrak{p}(e,1)$ contient des éléments nilpotents non nuls.

Proposition 1.4.1. *Les composantes irréductibles de $\mathfrak{C}^{nil}(\mathfrak{p})$ sont de la forme $\mathfrak{C}(e)$ avec e presque $\mathfrak{p}$-distingué.*

Démonstration. Choisissons e et $j \in I_e$ tels que $\mathfrak{C}(e)^{(j)}$ soit une composante irréductible de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g})$ et supposons que e ne soit pas presque $\mathfrak{p}$-distingué. Alors $\mathcal{N} \cap \mathfrak{p}(e,0)$ est non-trivial et $\mathcal{N}(\mathfrak{p}(e,0))_j$ contient un élément non-nul e_0. Par définition de $\mathcal{N}_e^{(j)}$, on a $e + e_0 \in \mathcal{N}_e^{(j)}$, donc $e + e_0 \in \overline{\mathcal{O}(e)}$ par le lemme 1.4.2. Si $e = 0$ on a une contradiction, on suppose donc $e \neq 0$.

Comme $\mathfrak{g}(e,0)$ est réductive dans $\mathfrak{g}$, on peut inclure e_0 dans un $\mathfrak{sl}_2$-triplet normal (e_0, h_0, f_0) de $\mathfrak{g}(e,0)$. Soit $\mathfrak{s}_0$ l'algèbre de Lie de dimension trois engendrée par (e_0, h_0, f_0). Comme $\mathfrak{s}_0 \subseteq \mathfrak{g}(e,0)$, $(e + e_0, h + h_0, f + f_0)$ est un $\mathfrak{sl}_2$-triplet normal. Maintenant, $e \in \mathfrak{g}(h_0, 0)$, $e_0 \in \mathfrak{g}(h_0, 2)$ donc l'application $\tau_\lambda \in \mathrm{Aut}(\mathfrak{g})$ $(\lambda \in \Bbbk^*)$ définie dans [TY, 38.6.2] envoie $e + e_0$ sur $e + \lambda^2 e_0$. On en déduit que $e + \Bbbk^* e_0 \subset \mathcal{O}(e + e_0)$ et donc $e \in \overline{\mathcal{O}(e + e_0)}$. Rappelons que d'après le paragraphe précédent, on a $e + e_0 \in \overline{\mathcal{O}(e)}$. On en déduit que $e + e_0$ et e sont K-conjugués. Ceci implique que h et $h + h_0$ sont également K-conjugués. On va maintenant chercher une contradiction.

On a $L(h, h_0) = L(h, [e_0, f_0]) = L([e_0, h], f_0) = 0$ où L désigne la forme de Killing de $\mathfrak{g}$. Ceci implique que $L(h + h_0, h + h_0) = L(h,h) + L(h_0, h_0)$. Mais h et $h + h_0$ sont conjugués et L est invariante sous l'action de G, d'où $L(h_0, h_0) = 0$. Or, puisque h_0 est l'élément semi-simple d'un $\mathfrak{sl}_2$-triplet, l'action adjointe de h_0 est à valeurs propres entières, et donc $L(h_0, h_0) = 0$ si et seulement si $h_0 = 0$. Ceci est impossible par le choix de e_0. On a donc montré par l'absurde que e est presque $\mathfrak{p}$-distingué.

Maintenant, comme $\mathfrak{p}(e,0)$ ne contient pas d'éléments nilpotents et que $\mathcal{N}\cap\mathfrak{p}^e = \bigoplus_{i>0}\mathfrak{p}(e,i)$ est irréductible, on en déduit que $\mathfrak{C}(e) = \mathfrak{C}(e)^{(j)}$ est la composante irréductible de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$ engendrée par e. □

Remarque 1.4.4. Notons que d'après la preuve de la proposition précédente $\mathfrak{C}(e)$ est irréductible pour e presque $\mathfrak{p}$-distingué. On a donc $\mathfrak{C}(e) = \overline{K.(e,\mathcal{N}\cap\mathfrak{p}^e)}$ lorsque e est presque $\mathfrak{p}$-distingué et la présence des indices $j \in I_e$ n'est plus nécessaire.

Corollaire 1.4.5. *Les composantes étranges sont engendrées par des éléments presque $\mathfrak{p}$-distingués non $\mathfrak{p}$-distingués.*

D'après la proposition 1.3.1 et le corollaire 1.4.5 la conjecture A est équivalente au fait que $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$ ne comporte pas de composantes étranges, ou encore :

Conjecture 1.4.2. *Si e_1 est presque $\mathfrak{p}$-distingué, il existe un élément e_2 $\mathfrak{p}$-distingué tel que $\mathfrak{C}(e_1) \subseteq \mathfrak{C}(e_2)$.*

D'après les résultats de la section 1.1, cette conjecture est vraie si et seulement si elle est vraie pour toute algèbre de Lie symétrique simple.

2 Centralisateurs

Le but de cette section est de caractériser les centralisateurs de $\mathfrak{sl}_2$-triplets dans le cas classique pour obtenir une classification des éléments presque $\mathfrak{p}$-distingués. En particulier nous allons retrouver que les éléments de défaut nul (*i.e.* $\mathfrak{p}$-distingués) correspondent bien aux éléments compacts décrits dans [PT]. Le travail de [PT] donne une description des éléments $\mathfrak{p}$-distingués et utilise le fait que les orbites d'éléments compacts dans le cas réel correspondent, via la correspondance de Kostant-Sekiguchi, aux orbites d'éléments $\mathfrak{p}$-distingués. Notre démarche pour décrire les centralisateurs lui sera en fait assez similaire. Dans toute la section, V désignera un espace vectoriel de dimension n. On supposera que $\mathfrak{g} \subset \mathfrak{sl}(V)$ est une algèbre de Lie simple et que, contrairement à l'introduction, $G \subset \mathrm{SL}(V)$ est le plus petit groupe algébrique dont l'algèbre de Lie contient $\mathfrak{g}$ (cf. [TY, Chapitre 24]). Cette modification n'a pas d'incidence sur les orbites de $\mathfrak{g}$. On se restreint aux cas classiques, c'est à dire que l'on est dans l'une des situations suivantes :

- Type A : $\mathfrak{g} = \mathfrak{sl}(V)$ auquel cas $G = SL(V)$.
- Type BD : $\mathfrak{g} = \mathfrak{so}_\Phi(V) = \mathrm{End}(V,\Phi)$ est l'algèbre de Lie orthogonale stabilisant une forme bilinéaire symétrique non dégénérée Φ. Dans ce cas $G = SO_\Phi(V)$.
- Type C : $\mathfrak{g} = \mathfrak{sp}_\Phi(V) = \mathrm{End}(V,\Phi)$ est l'algèbre de Lie symplectique stabilisant une forme bilinéaire alternée non dégénérée Φ. Dans ce cas $G = SP_\Phi(V)$.

Définition 2.0.1. [GW, Théorème 3.4] Enumérons l'ensemble des involutions de Cartan θ définies sur $\mathfrak{g}$ dans chacun de ces types.

- Type A :

(AI) $\theta(x) = -T{}^tx{}^tT$ pour $x \in \mathfrak{g}$ et où $T = T^{-1} = {}^tT$. Ceci est équivalent au fait que $\mathfrak{k} = \mathrm{End}(V, \Phi)$ où $\Phi(x, y) = {}^txTy$ est la forme bilinéaire symétrique associée à T. La propriété $T = T^{-1} = {}^tT$ définit de façon unique θ (à conjugaison près dans G).

(AII) $\theta(x) = -T{}^tx{}^tT$ pour $x \in \mathfrak{g}$ et où $T = T^{-1} = -{}^tT$. Ceci est équivalent au fait que $\mathfrak{k} = \mathrm{End}(V, \Phi)$ où $\Phi = {}^txTy$ est la forme bilinéaire antisymétrique associée à T. La propriété $T = T^{-1} = -{}^tT$ définit de façon unique θ (à conjugaison près dans G).

(AIII) $\theta(x) = JxJ^{-1}$ pour $x \in \mathfrak{g}$ et où $J^2 = \mathrm{Id}_V$. On définit

$$V_\pm = \{v \in V \mid Jv = \pm v\}.$$

Alors $V = V_+ \oplus V_-$ et le nombre $\dim(V_+)$ définit θ de façon unique (à conjugaison près dans G).

- Types BD et C :

(BDI), (CII) $\theta(x) = JxJ^{-1}$ pour $x \in \mathfrak{g}$ où J préserve la forme Φ et vérifie $J^2 = \mathrm{Id}_V$. On définit

$$V_\pm = \{v \in V \mid Jv = \pm v\}.$$

Alors $V = V_+ \oplus V_-$, la restriction de Φ à $V_\pm$ est non-dégénérée et le nombre $\dim(V_+)$ définit θ de façon unique (à conjugaison près dans G).

(DIII), (CI) $\theta(x) = JxJ^{-1}$ pour $x \in \mathfrak{g}$ où J préserve la forme Φ et vérifie $J^2 = -\mathrm{Id}_V$. On définit

$$V_{\pm i} = \{v \in V \mid Jv = \pm iv\}.$$

Alors $V = V_+ \oplus V_-$, la restriction de Φ à $V_{\pm i}$ est nulle et V_{+i} dual à V_{-i} par rapport à ω. De plus l'involution θ est uniquement déterminée (à conjugaison près dans G).

Etant donnée une telle algèbre de Lie symétrique $(\mathfrak{g}, \mathfrak{k})$ ou $(\mathfrak{g}, \theta)$ et $e \in \mathfrak{p}$ nilpotent non nul, on fixe un $\mathfrak{sl}_2$-triplet (e, h, f) normal qui engendre une sous-algèbre de Lie de dimension 3 que l'on note $\mathfrak{s}$. Les centralisateurs $\mathfrak{g}^{\mathfrak{s}}$ sont connus, cf. [SS] ou [Ja]. Afin d'obtenir des informations précises sur les paires symétriques $(\mathfrak{g}^{\mathfrak{s}}, \theta_{|\mathfrak{g}^{\mathfrak{s}}})$, nous allons rappeler la description des sous-algèbres $\mathfrak{g}^{\mathfrak{s}}$.

Pour $\lambda \in \mathbb{Z}$ notons :

$$V(\lambda) = \{v \in V \mid h.v = \lambda v\}$$

de sorte que $V = \bigoplus_{\lambda \in \mathbb{Z}} V(\lambda)$. Il existe (à isomorphisme près) un unique $\mathfrak{sl}_2$-module irréductible de dimension d. Notons-le ρ_d. Alors $V = \bigoplus_{d \in \mathbb{N}^*} V_d$ où V_d est isomorphe à $m_d\, \rho_d$. Ceci nous donne une décomposition

$$V_d = \bigoplus_{j \in [\![0, d-1]\!]} V(d - 1 - 2j) \cap V_d. \tag{2.1}$$

On définit alors $V_{j,d} := V(d-1-2j) \cap V_d$ et $m_d := \dim V_{j,d}$, ce dernier étant un entier indépendant de j. On en déduit une partition de n donnée par $(d^{m_d})_{d\in\mathbb{N}^*}$. Cette partition correspond au diagramme de Young usuellement associé à la classe de conjuguaison d'un élément nilpotent de $\mathfrak{sl}(V)$ (cf. [Oh1], par exemple). Notons H_d (resp. H'_d) le sous-espace vectoriel engendré par les vecteurs de plus haut (resp. bas) poids de V_d. Autrement dit $H_d = V_{0,d}$ et $H'_d = V_{d-1,d}$. Soit g un élément de $\mathfrak{gl}(V)^{\mathfrak{s}}$. Par définition il stabilise les sous espaces propres de h et de e, et en particulier g stabilise les sous-espaces $H_d = V(d-1) \cap \ker e$. On peut donc définir des applications restrictions indexées par $d \in \mathbb{N}^*$

$$\varphi_d : \left\{ \begin{array}{rcl} \mathfrak{gl}(V)^{\mathfrak{s}} & \longrightarrow & \mathfrak{gl}(H_d) \\ g & \longmapsto & g_{|H_d}. \end{array} \right.$$

Et on définit

$$\varphi = \bigoplus_{d\in\mathbb{N}^*} \varphi_d \tag{2.2}$$

On peut alors énoncer un premier résultat sur les centralisateurs :

Lemme 2.0.2. *Le centralisateur du $\mathfrak{sl}_2$-triplet (e, h, f) dans $\mathfrak{gl}(V)$ vérifie :*

$$\mathfrak{gl}(V)^{\mathfrak{s}} \cong^{\varphi} \bigoplus_{d\in\mathbb{N}^*} \mathfrak{gl}(H_d).$$

Démonstration. On a vu que l'application φ est bien définie. Nous allons démontrer qu'elle possède une réciproque. Soit $(g_d)_{d\in\mathbb{N}^*}$ une famille d'éléments de $\bigoplus_{d\in\mathbb{N}^*} \mathfrak{gl}(H_d)$, nous allons montrer qu'il existe un unique élément $g \in \mathfrak{gl}(V)$ tel que g centralise $\mathfrak{s}$ et $g_{|H_d} = g_d$. La formule (2.1) associée au fait que f^j induit une bijection entre H_d et $V_{j,d}$ nous montre qu'un tel élément g est nécessairement défini uniquement sur chaque composante de V_d par

$$g_{|V_{j,d}} = f^j.g_d.f^{-j}. \tag{2.3}$$

Comme V est somme directe des $V_{j,d}$, l'élément g est défini uniquement sur V tout entier. Réciproquement, il est facile de vérifier qu'un élément g, défini par (2.3) à partir d'une famille $(g_d)_{d\in\mathbb{N}^*}$ quelconque, commute avec $\mathfrak{s}$. $\square$

A partir de maintenant, on se donne une forme bilinéaire non-dégénérée Φ sur V et deux éléments $\varepsilon, \eta \in \{\pm 1\}$ de sorte que

- Φ est symétrique ou antisymétrique ce qui se traduit par $\Phi(u, v) = \varepsilon\Phi(v, u)$ pour tout $u, v \in V$.
- h préserve Φ et e, f préservent ou anti-préservent Φ : c'est à dire

$$\Phi(h.u, v) = -\Phi(u, h.v) \ ; \ \Phi(e.u, v) = -\eta\Phi(u, e.v) \ ; \ \Phi(f.u, v) = -\eta\Phi(u, f.v).$$

Le cas $\eta = 1$ permettra de décrire $\mathfrak{g}^{\mathfrak{s}}$ quand $\mathfrak{g} = \mathrm{End}(V, \Phi)$, tandis que le cas $\eta = -1$ permettra de décrire $\mathfrak{k}^{\mathfrak{s}}$ quand $\mathfrak{k} = \mathfrak{g} \cap \mathrm{End}(V, \Phi)$.

Lemme 2.0.3.

(a) Si $\lambda \neq -\mu$ alors $\Phi(V(\lambda), V(\mu)) = 0$. Les sous-espaces $V(\lambda)$ et $V(-\lambda)$ sont en dualité par Φ.

(b) H_d est dual à $H'_d = f^{d-1}(H_d)$.

Démonstration. Soient $u \in V(\lambda), v \in V(\mu)$.

$$\begin{aligned} \lambda\,\Phi(u,v) &= \Phi(h.u,v) \\ &= -\Phi(u,h.v) = -\mu\,\Phi(u,v), \end{aligned}$$

ce qui démontre la partie (a).
Soit maintenant $u \in H_d$. Par (a), il existe $w \in V(-d+1)$ tel que $\Phi(u,w) \neq 0$. Il existe w_1, w_2 tels que $w = e.w_1 + w_2$ et $w_2 \in H'_d$. On a alors :

$$0 \neq \Phi(u,w) = \Phi(u, e.w_1 + w_2) = -\eta\Phi(e.u, w_1) + \Phi(u, w_2) = \Phi(u, w_2).$$

Ceci prouve que la restriction de Φ à $H_d \times H'_d$ est non dégénérée. □

Définition 2.0.4. On définit $\tilde{\Phi}_d$ sur H_d par

$$\tilde{\Phi}_d(u,v) = \Phi(u, f^{d-1}.v).$$

Lemme 2.0.5. *La forme $\tilde{\Phi}_d$ est bilinéaire non dégénérée sur H_d et elle vérifie $\tilde{\Phi}_d(u,v) = (-\eta)^{d-1}\varepsilon\,\tilde{\Phi}_d(v,u)$.*

Démonstration. L'application f^{d-1} est une bijection de H'_d sur H_d, donc par le lemme 2.0.3 (b), $\tilde{\Phi}_d$ est non-dégénérée sur H_d. La relation de symétrie est une vérification facile. □

Nous pouvons maintenant énoncer un second résultat sur les centralisateurs.

Proposition 2.0.3. *Si l'algèbre de Lie $\mathfrak{w} \subseteq \mathfrak{sl}(V)$ est l'algèbre préservant la forme Φ sur V alors le centralisateur de (e,h,f) dans $\mathfrak{w}$ vérifie*

$$\mathfrak{w}^{\mathfrak{s}} \cong^{\varphi} \bigoplus_{d \in \mathbb{N}^*} \mathrm{End}(H_d, \tilde{\Phi}_d).$$

Démonstration. Par le lemme 2.0.2, on sait que tout élément de $\mathfrak{w}^{\mathfrak{s}}$ peut être vu comme un élément de $\bigoplus_{d \in \mathbb{N}^*} \mathfrak{gl}(H_d)$. Maintenant, le fait que $g \in \mathfrak{w}$ implique que pour tout $d \in \mathbb{N}^*$, $(u,w) \in H_d \times H'_d$, on a $\Phi(g.u, w) = -\Phi(u, g.w)$. C'est à dire que pour tout $(u,v) \in H_d$, on a $0 = \Phi(g.u, f^{d-1}.v) + \Phi(u, gf^{d-1}.v) = \tilde{\Phi}_d(g.u, v) + \tilde{\Phi}_d(u, g.v)$, ce qu'on peut reformuler en : $g_{|H_d}$ préserve $\tilde{\Phi}_d$.

Réciproquement, si g est défini à partir d'éléments $g_d \in \mathrm{End}(H_d, \tilde{\Phi}_d)$ et de la formule (2.3), alors g stabilise les sous-espaces $V_{j,d}$ de (2.1). Il suffit donc de regarder g sur tous les éléments

$(u, v) \in V_{j,d} \times V_{d-1-j,d}$ pour montrer que g préserve Φ :

$$\begin{aligned}\Phi(g.u, v) = \Phi(f^j g_d f^{-j}.u, v) &= (-\eta)^j \Phi(g_d f^{-j}.u, f^j.v)\\ &= (-\eta)^j\, \tilde{\Phi}(g_d f^{-j}.u, f^{j-(d-1)}.v)\\ &= -(-\eta)^j\, \tilde{\Phi}(f^{-j}.u, g_d f^{j-(d-1)}.v)\\ &= -(-\eta)^j\, \Phi(f^{-j}.u, f^{d-1} g_d f^{j-(d-1)}.v)\\ &= -\Phi(u, f^{d-1-j} g_d f^{j-(d-1)}.v) = -\Phi(u, g.v).\end{aligned}$$

□

Dorénavant J désigne un élément de $GL(V)$ tel qu'il existe $\xi \in \{\pm 1\}$ avec $J^2 = \xi\,\mathrm{Id}$. Pour $g \in \mathfrak{gl}(V)$, on note $\theta(g) = JgJ^{-1}$. On suppose de plus que h commute avec J, et e, f anti-commutent avec J. C'est à dire :

$$\theta(h) = h\ ;\ \theta(e) = -e\ ;\ \theta(f) = -f$$

On note $\sqrt{-1}$ une racine carrée de -1 dans $\Bbbk$.

Lemme 2.0.6.

(a) Le sous-espace H_d est J-stable.

(b) Si J préserve Φ, alors la restriction de $(\sqrt{-1})^{d-1}J$ à H_d préserve la forme $\tilde{\Phi}$.

Démonstration. Les endomorphismes J et h commutent, donc le h-sous-espace propre $V(d-1)$ est stable par J. De la même façon, J et e anti-commutent donc le noyau de e est stable par J. On en déduit la partie (a) en remarquant que $H_d = V(d-1) \cap \ker e$.

Si J préserve Φ, alors pour tout $u, v \in H_d$,

$$\begin{aligned}\tilde{\Phi}((\sqrt{-1})^{d-1}J.u, (\sqrt{-1})^{d-1}J.v) &= (-1)^{d-1}\Phi(J.u, f^{d-1}J.v)\\ &= (-1)^{2(d-1)}\Phi(J.u, Jf^{d-1}.v)\\ &= \Phi(u, f^{d-1}v) = \tilde{\Phi}(u, v).\end{aligned}$$

□

Remarque 2.0.7. Pour tout $d \in \mathbb{N}^*$, l'involution θ est inchangée par la modification de J en $(\sqrt{-1})^{d-1}J$. En effet, pour tout $g \in \mathfrak{gl}(V)$,
$(\sqrt{-1})^{d-1}Jg((\sqrt{-1})^{d-1}J)^{-1} = JgJ^{-1}$. Par contre, ce qui change ce sont les formes préservées par J et $(\sqrt{-1})^{d-1}J$.

Corollaire 2.0.8. *Si $\mathfrak{gl}^{\theta}(V) \subseteq \mathfrak{gl}(V)$ est l'algèbre de Lie (réductive) des points fixes de θ alors*

$$\mathfrak{gl}^{\theta}(V)^{\mathfrak{s}} \cong^{\varphi} \bigoplus_{d \in \mathbb{N}^*} \mathfrak{gl}^{\theta_d}(H_d)$$

où $\theta_d(h)$ pour $h \in H_d$ est donné par $J_{|H_d} h J^{-1}_{|H_d}$.

Démonstration. Le seul point restant non-trivial consiste à vérifier qu'un élément g déterminé à partir d'éléments g_d comme dans la formule (2.3) est bien invariant par θ. En effet, si $u \in V_{j,d}$:

$$\theta(g).u = \theta(f^j g_d f^{-j}).u = (-1)^{j-j} f^j \theta(g_d) f^{-j}.u = g.u$$

□

Enfin, cette dernière proposition nous permettra de traiter les derniers centralisateurs classiques envisageables.

Proposition 2.0.4. *Si J préserve Φ et que $\mathfrak{k} = \mathrm{End}(V, \Phi)^J \subseteq \mathfrak{gl}(V)$ est l'algèbre de Lie réductive donnée par l'intersection de $\mathfrak{gl}^\theta(V)$ et de $\mathrm{End}(V, \Phi)$, alors*

$$\mathfrak{k}^s \cong^\varphi \bigoplus_{d \in \mathbb{N}^*} (\mathrm{End}(H_d, \tilde{\Phi}))^{(\sqrt{-1})^{d-1} J}.$$

Démonstration. Il suffit de combiner le corollaire 2.0.8 et la proposition 2.0.3. □

3 Classification des éléments presque $\mathfrak{p}$-distingués dans le cas classique

Reprenons les notations de la section 2 où $\mathfrak{g} \subseteq \mathfrak{sl}(V) = \mathfrak{sl}_n$ désignera une algèbre de Lie classique simple (cf. 2.0.1), θ une involution non-triviale sur $\mathfrak{g}$ et $\mathfrak{k}, \mathfrak{p}$ les sous-espaces propres de θ associés respectivement aux valeurs propres $+1$ et -1. On supposera toujours que si $e \in \mathfrak{p}$ est un élément nilpotent non nul, on l'inclut dans un $\mathfrak{sl}_2$-triplet normal (e, h, f). On laisse au lecteur le soin de traduire les assertions quand $e = 0$. Rappelons, cf. section précédente, qu'à e est associée canoniquement la partition $(d^{m_d})_{d \in \mathbb{N}^*}$ de n. On notera ces partitions sous la forme d'un diagramme de Young, qu'on appellera diagramme de Young associé à e. Dans le cas où une matrice J intervient dans la définition de l'involution sur l'algèbre de Lie (cf. 2.0.1), on introduit la notion d'ab-diagramme de Young associé à e (cf. par exemple [Oh1]) de la façon suivante. On décompose V en J-sous-espaces propres $V_a \oplus V_b$ et on fixe une base diagonalisant J

$$\{\alpha_{d,i}^j \mid d \in \mathbb{N}^*, i \in [\![0, d-1]\!], j \in [\![1, m_d]\!]\}$$

adaptée au $\mathfrak{sl}_2$-triplet, c'est à dire telle que $(\alpha_{d,0}^j)_j$ forme une base de H'_d et $e(\alpha_{d,i}^j) = \alpha_{d,i+1}^j$. On remplit alors les cases du diagramme de Young associé à e par a ou b suivant que $\alpha_{d,i}^j \in V_a$ ou V_b. Comme $\theta(e) = -e$, sur une ligne du diagramme de Young, on a alternance de a et de b. On définit aussi $H'_{d,a} = V_a \cap H'_d$ (resp. $H'_{d,b} = V_b \cap H'_d$) et $a_d = \dim H'_{d,a}$ (resp. $b_d = \dim H'_{d,b}$) de sorte que $a_d + b_d = m_d$ et que a_d (resp b_d) désigne le nombre de lignes de longueur d commençant par un a (resp. b). Les éléments $(a_d)_{d \in \mathbb{N}^*}$ et $(b_d)_{d \in \mathbb{N}^*}$ sont des invariants de la K-classe de conjugaison de e. Lorsque J n'intervient pas dans la définition de l'involution (type AI et AII), le diagramme de Young ou la donnée de $(m_d)_{d \in \mathbb{N}^*}$ sera un invariant suffisant.

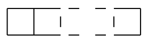
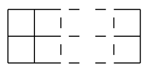
Définition 3.0.1. On appelle (ab-)diagramme de Young, soit un diagramme de Young (types AI et AII), soit un ab-diagramme de Young (autres types classiques) associé à une orbite $\mathcal{O}(e)$ comme ci-dessus.

Rappelons qu'en vertu de la section précédente, on a des décompositions θ-stables, indexées par d grâce à l'application φ, des algèbres de Lie $\mathfrak{gl}(V), \mathfrak{so}(V)$ et $\mathfrak{sp}(V)$. La définition suivante utilise ces décompositions.

Définition 3.0.2. On appelle défaut de la longueur d et on note $\delta^e(d)$ le rang de la paire symétrique $(\mathfrak{g}'_d, \mathfrak{k}_d) := (\varphi_d(\mathfrak{g}'), \varphi_d(\mathfrak{k}))$ où $\mathfrak{g}' = \mathfrak{g}$ dans les cas DIII, BDI, CI et CII et $\mathfrak{g}' = \mathfrak{gl}(V)$ dans les cas AI, AII et AIII.

Remarque 3.0.3. Sous les notations précédentes, on a $\delta(e) = \sum_d \delta^e(d)$ dans les cas DIII, BDI, CI et CII et $\delta(e) = \sum_d \delta^e(d) - 1$ dans les cas AI, AII et AIII.

Les sous-sections 3.1, 3.2 et 3.3 donnent les (ab-)diagrammes de Young des éléments $\mathfrak{p}$-distingués et presque $\mathfrak{p}$-distingués pour chacun des cas présentés dans la définition 2.1. On rappelle que dans chaque cas simple classique, les (ab-)diagrammes possibles sont les concaténations des (ab-)diagrammes primitifs que nous récapitulons dans le tableau suivant tiré des travaux de T. Ohta. Rappelons aussi que les (ab-)diagrammes sont classiquement ordonnés de façon décroissante de la plus grande ligne en haut vers la plus petite en bas et qu'on les regarde à permutation près de lignes de même longueur.

Type	(ab-)diagrammes primitifs		
AI			
AII			
AIII	$ab....ba$, $ba....ab$, $ab....ab$, $ba....ba$,		
BDI	$ab....ba$,	$ba....ab$,	$ab....ab$ $ba....ba$,
CI	$ab....ab$,	$ba....ba$,	$ab....ba$ $ba....ab$,
DIII	$ab....ab$ $ab....ab$,	$ba....ba$ $ba....ba$,	$ab....ba$ $ba....ab$,
CII	$ab....ba$ $ab....ba$,	$ba....ab$ $ba....ab$,	$ab....ab$ $ba....ba$,

3.1 Cas AI et AII

La paire $(\mathfrak{g}, \mathfrak{k})$ considérée est de la forme suivante : $(\mathfrak{sl}(V), \mathrm{End}(V, \Phi))$ où Φ est une forme bilinéaire symétrique (AI) ou anti-symétrique (AII). On pose $\mathfrak{g}' = \mathfrak{gl}(V)$ équipée de l'involution définie en 2.0.1 (cas AI et AII). On a alors $\mathfrak{g}' = \mathfrak{g} + \Bbbk\mathrm{Id}_n$ où $\Bbbk\mathrm{Id}_n \subset \mathfrak{p}'$. D'après le lemme 2.0.2, on a $\mathfrak{g}'^{\mathfrak{s}} \cong \bigoplus_d \mathfrak{gl}_{m_d}$. Et à l'intérieur de ce centralisateur, par la proposition 2.0.3 avec $\mathfrak{w} = \mathfrak{k}$ et $\eta = -1$, on obtient $\mathfrak{k}^{\mathfrak{s}} \cong \bigoplus_d \mathfrak{so}_{m_d}$ (AI) ou $\mathfrak{k}^{\mathfrak{s}} \cong \bigoplus_d \mathfrak{sp}_{m_d}$ (AII). Pour $d \in \mathbb{N}$ fixé, la paire réductive symétrique associée est donc une paire réductive du même type que celle de l'algèbre de Lie symétrique de départ. On en déduit que $\delta^e(d) = m_d$ (AI) ou $\delta^e(d) = \frac{m_d}{2}$ (AII). Regardons maintenant dans quels cas $(\mathfrak{g}'^{\mathfrak{s}} = \mathfrak{k}^{\mathfrak{s}} \oplus T_r)$ où T_r est un tore de dimension $r \geqslant 1$, auquel cas e est presque distingué de défaut $r-1$. Il suffit de regarder les sous-algèbres $(\mathfrak{g}'_d, \mathfrak{k}_d)$.

AI :
- Si $m_d = 1$, la paire réductive associée est $(T_1, \{0\})$ avec $\delta^e(d) = 1$,
- Si $m_d \geqslant 2$, il existe un élément nilpotent non nul pour la paire $(\mathfrak{gl}_{m_d}, \mathfrak{so}_{m_d})$. Cet élément fournit un élément nilpotent non nul pour la paire $(\mathfrak{gl}_n, \mathfrak{so}_n)$ qui est également nilpotent dans $(\mathfrak{sl}_n, \mathfrak{so}_n)$.

AII : Les éléments m_d sont pairs.
- Si $m_d = 2$, la paire réductive associée est $(\mathfrak{gl}_2, \mathfrak{sl}_2)$ où $\mathfrak{gl}_2 \cong \mathfrak{sl}_2 \oplus T_1$.
- Si $m_d \geqslant 4$, il existe des éléments nilpotents non nuls pour $(\mathfrak{g}'_d, \mathfrak{k}_d)$. Ces éléments correspondent à des éléments nilpotents non nuls de $(\mathfrak{g}, \mathfrak{k})$.

On en déduit la proposition suivante :

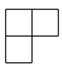
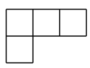
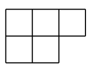
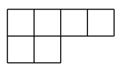
Proposition 3.1.1. *Les éléments $\mathfrak{p}$-distingués dans le cas AI et AII sont les éléments $\mathfrak{p}$-réguliers. Les éléments presque $\mathfrak{p}$-distingués du cas AI (resp. AII) sont ceux qui correspondent à un diagramme de Young ayant des lignes de longueurs distinctes (resp. des paires de lignes de longueurs distinctes). Le défaut d'un élément nilpotent est égal respectivement, au nombre de lignes ou de paires de lignes, moins une.*

Démonstration. L'assertion sur le défaut provient de la formule

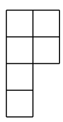
$$\delta(e) = \sum_d \delta^e(d) - 1$$

de la remarque 3.0.3. Celle sur les éléments $\mathfrak{p}$-distingués provient de la discussion de cas ci-dessus. Enfin, il est connu que les éléments $\mathfrak{p}$-réguliers dans le cas AI (resp. AII) correspondent aux diagrammes de Young constitués d'une seule ligne (resp. paire de lignes). □

Exemple 3.1.1. Dans le cas AI, les éléments correspondant aux diagrammes suivants sont presque $\mathfrak{p}$-distingués de défaut 1. En particulier, ils ne sont pas $\mathfrak{p}$-distingués :

$\Gamma_1 =$  , $\Gamma'_1 =$  , $\Gamma_2 =$  , $\Gamma_3 =$  .

Dans le cas AII, ce sont les mêmes en doublant les lignes, par exemple :

$\Gamma_4 =$ 

3.2 Cas AIII

La paire $(\mathfrak{g}, \mathfrak{k})$ considérée est de la forme suivante : $(\mathfrak{sl}_n, \mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus T_1)$ où $p+\ q\ =\ n$. D'après le lemme 2.0.2, on a $\mathfrak{g}^\mathfrak{s} \cong (\bigoplus_d \mathfrak{gl}_{m_d}) \cap \mathfrak{sl}_n$. Et à l'intérieur de ce centralisateur, par la proposition 2.0.8, on obtient $\mathfrak{k}^\mathfrak{s} \cong \bigoplus_d (\mathfrak{gl}_{a_d} \oplus \mathfrak{gl}_{b_d}) \cap \mathfrak{sl}_{m_d}$. Pour $d \in \mathbb{N}$ fixé, la paire réductive symétrique associée est donc une paire du même type que celle de l'algèbre de Lie symétrique de départ. Regardons maintenant dans quels cas $(\mathfrak{g}^\mathfrak{s} = \mathfrak{k}^\mathfrak{s} \oplus T_r)$ où T_r est un tore de dimension $r \geqslant 0$.

- Si $a_d b_d = 0$, la paire associée est $(\mathfrak{sl}_n, \mathfrak{sl}_n)$.
- Si $a_d \neq 0$ et $b_d \neq 0$, il existe un élément nilpotent non nul pour la paire $(\mathfrak{gl}_{m_d}, (\mathfrak{gl}_{a_d} \oplus \mathfrak{gl}_{b_d}))$. C'est un élément nilpotent pour la paire $(\mathfrak{sl}_n, \mathfrak{sl}_p\ \oplus\ \mathfrak{sl}_q\ \oplus T_1)$.

On en déduit la proposition suivante :

Proposition 3.2.1. *Les éléments $\mathfrak{p}$-distingués dans le cas AIII coïncident avec les éléments presque $\mathfrak{p}$-distingués. Ce sont ceux qui correspondent aux ab-diagrammes de Young dont les lignes de longueur fixée commencent par la même lettre.*

3.3 Cas BDI, CI, DIII et CII

La paire $(\mathfrak{g}, \mathfrak{k})$ considérée est de la forme suivante : $(\mathrm{End}(V, \Phi), \mathrm{End}(V, \Phi)^J)$ où Φ est une forme bilinéaire non-dégénérée symétrique ($\varepsilon = 1$) (BD) ou antisymétrique ($\varepsilon = -1$) (C) ; J préserve Φ et vérifie $J^2 = \xi \mathrm{Id}$ avec $\xi = 1$ (BDI, CII) ou $\xi = -1$ (DIII,CI).

D'après la proposition 2.0.3, on a $\mathfrak{g}^\mathfrak{s} \cong \bigoplus_d \mathrm{End}(H_d, \tilde{\Phi}_d)$. A l'intérieur de ce centralisateur, par la proposition 2.0.4, on obtient

$$\mathfrak{k}^s \cong \bigoplus_{d \in \mathbb{N}^*} (\mathrm{End}(H_d, \tilde{\Phi}_d))^{(\sqrt{-1})^{d-1} J}.$$

On pose $\varepsilon_d = 1$ si $\tilde{\Phi}_d$ est symétrique et $\varepsilon_d = -1$ si $\tilde{\Phi}_d$ est anti-symétrique. De même, on définit ξ_d par $((\sqrt{-1})^{d-1} J_{|H_d})^2 = \xi_d \mathrm{Id}$. Par les lemmes 2.0.5 et 2.0.6 (b) où l'on pose $\eta = 1$, on a $(\varepsilon_d, \xi_d) = ((-1)^{d-1}\varepsilon, (-1)^{d-1}\xi)$. Notons que les sous-espaces propres de $(\sqrt{-1})^{d-1} J$ sont de dimensions a_d et b_d ; si $\xi_d = -1$ alors nécessairement $a_d = b_d = \frac{m_d}{2}$. Pour $d \in \mathbb{N}$ fixé, la paire réductive symétrique associée est donc donnée par le tableau suivant :

$(\mathfrak{g}, \mathfrak{k})$	(ε, ξ)	d pair		d impair	
		(ε_d, ξ_d)	$(\mathfrak{g}_d, \mathfrak{k}_d)$	(ε_d, ξ_d)	$(\mathfrak{g}_d, \mathfrak{k}_d)$
$(\mathfrak{so}_n, \mathfrak{so}_p \times \mathfrak{so}_q)$ (BDI)	(1,1)	(-1,-1)	$(\mathfrak{sp}_{m_d}, \mathfrak{gl}_{\frac{m_d}{2}})$	(1,1)	$(\mathfrak{so}_{m_d}, \mathfrak{so}_{a_d} \times \mathfrak{so}_{b_d})$
$(\mathfrak{sp}_n, \mathfrak{gl}_{\frac{n}{2}})$ (CI)	(-1,-1)	(1,1)	$(\mathfrak{so}_{m_d}, \mathfrak{so}_{a_d} \times \mathfrak{so}_{b_d})$	(-1,-1)	$(\mathfrak{sp}_{m_d}, \mathfrak{gl}_{\frac{m_d}{2}})$
$(\mathfrak{so}_n, \mathfrak{gl}_{\frac{n}{2}})$ (DIII)	(1,-1)	(-1,1)	$(\mathfrak{sp}_{m_d}, \mathfrak{sp}_{a_d} \times \mathfrak{sp}_{b_d})$	(1,-1)	$(\mathfrak{so}_{m_d}, \mathfrak{gl}_{\frac{m_d}{2}})$
$(\mathfrak{sp}_n, \mathfrak{sp}_p \times \mathfrak{sp}_q)$ (CII)	(-1,1)	(1,-1)	$(\mathfrak{so}_{m_d}, \mathfrak{gl}_{\frac{m_d}{2}})$	(-1,1)	$(\mathfrak{sp}_{m_d}, \mathfrak{sp}_{a_d} \times \mathfrak{sp}_{b_d})$

Regardons maintenant dans quels cas $(\mathfrak{g}^\mathfrak{s} = \mathfrak{k}^\mathfrak{s} \oplus T_r)$ où T_r est un tore de dimension $r \geqslant 0$.

(a) (BDI) Si $a_d b_d = 0$, alors $(\mathfrak{g}_d, \mathfrak{k}_d) = (\mathfrak{so}_{m_d}, \mathfrak{so}_{m_d})$.

(b) (BDI) Si $a_d b_d = 1$, on a $(\mathfrak{g}_d, \mathfrak{k}_d) = (T_1, \{0\})$.

(c) (BDI) Si $a_d b_d \geqslant 2$, $\mathfrak{p}_d$ contient des éléments nilpotents non nuls.

(d) (CI) Si $m_d \geqslant 2$ (i.e. $\neq 0$), $\mathfrak{p}_d$ contient des éléments nilpotents non nuls.

(e) (CII) Si $a_d b_d = 0$ alors $(\mathfrak{g}_d, \mathfrak{k}_d) = (\mathfrak{sp}_{m_d}, \mathfrak{sp}_{m_d})$.

(f) (CII) Si $a_d, b_d \geqslant 1$, $\mathfrak{p}_d$ contient des éléments nilpotents non nuls.

(g) (DIII) Si $m_d = 2$, on a $(\mathfrak{g}_d, \mathfrak{k}_d) = (T_1, T_1)$.

(h) (DIII) Si $m_d \geqslant 4$, $\mathfrak{p}_d$ contient des éléments nilpotents non nuls.

On en déduit les deux propositions suivantes :

Proposition 3.3.1. *Les éléments $\mathfrak{p}$-distingués du cas BDI (resp. CI) sont ceux qui ont un ab-diagramme n'ayant pas de lignes de longueur paire (resp. impaire) et tels que les lignes de longueur impaire (resp. paire) débutent toutes par la même lettre. Pour obtenir les éléments presque $\mathfrak{p}$-distingués, on autorise de plus, par longueur de ligne impaire (resp. paire) une paire de ligne, débutant l'une par a et l'autre par b. Le nombre de tels couples est égal au défaut de notre élément.*

Démonstration. Dans le cas (a), on a $\delta^e(d) = 0$. Dans le cas (b), on a $\delta^e(d) = \dim \mathfrak{p}_d^{\mathfrak{s}} = 1$. Dans les cas (c) et (d), on a des éléments nilpotents non nuls pour la paire $(\mathfrak{g}^{\mathfrak{s}}, \mathfrak{k}^{\mathfrak{s}})$, ce qui empêche l'élément d'être presque $\mathfrak{p}$-distingué. On en déduit la proposition grâce au tableau précédent. □

Proposition 3.3.2. *Les éléments $\mathfrak{p}$-distingués du cas CII (resp. DIII) sont ceux qui ont un ab-diagramme tels qu'il existe, par longueur de ligne paire (resp. impaire), une ou zéro paire de ligne (débutant nécessairement par a et par b) et tels que les lignes de longueur impaire (resp. paire) débutent toutes par la même lettre. Par ailleurs, les éléments presque $\mathfrak{p}$-distingués sont $\mathfrak{p}$-distingués.*

Démonstration. Dans les cas (e) et (g), on a $\delta^e(d) = 0$. Dans les cas (f) et (h), on a des éléments nilpotents non nuls pour la paire $(\mathfrak{g}^{\mathfrak{s}}, \mathfrak{k}^{\mathfrak{s}})$, ce qui empêche l'élément d'être presque $\mathfrak{p}$-distingué. On en déduit la proposition grâce au tableau précédent. □

Exemple 3.3.1. Dans le cas BDI, les éléments correspondant aux diagrammes suivants sont presque $\mathfrak{p}$-distingués de défaut 1. En particulier, ils ne sont pas distingués :

$$\Gamma_5 = \begin{array}{l} aba \\ a \\ b \end{array} , \qquad \Gamma_6 = \begin{array}{l} ababa \\ aba \\ bab \\ b \end{array} , \qquad \Gamma_7 = \begin{array}{l} ababa \\ aba \\ bab \\ a \end{array} .$$

Le cas CI est très similaire, des exemples de tels éléments sont les suivants :

$$\Gamma_8 = \begin{array}{l} baba \\ ba \\ ab \end{array}, \qquad \Gamma_9 = \begin{array}{l} bababa \\ baba \\ abab \\ ab \end{array}, \qquad \Gamma_{10} = \begin{array}{l} bababa \\ baba \\ abab \\ ba \end{array}.$$

3.4 Bilan Provisoire

A partir des examens de $(\mathfrak{g}'_d, \mathfrak{k}_d)$ dans certains cas classiques simples effectués lors de cette section, on peut résumer le défaut de longueur d associé à e dans le tableau ci-dessous.

Type	$\delta^e(d)$ pour $d \in \mathbb{N}^*$	
AI	m_d	
AII	$\frac{m_d}{2}$	
	d pair	d impair
(BDI)	$\frac{m_d}{2}$	$\min\{a_d, b_d\}$
(CI)	$\min\{a_d, b_d\}$	$\frac{m_d}{2}$

Par ailleurs, en combinant les résultats 3.2.1 et 3.3.2 avec la proposition 1.4.1, on obtient :

Corollaire 3.4.1. *La conjecture 1.4.2 est vraie dans les cas AIII, DIII et CII.*

4 Réduction d'orbites presque $\mathfrak{p}$-distinguées

Nous disposons d'un autre résultat pour écarter le fait que certains éléments presque $\mathfrak{p}$-distingués puissent engendrer une composante irréductible étrange.

Proposition 4.0.1. *(a) Soit $e_1, e_2 \in \mathfrak{p}$ deux éléments nilpotents avec $\mathcal{O}(e_1) \subset \overline{\mathcal{O}(e_2)}$ alors $\delta(e_1) - \delta(e_2) \leqslant \dim \mathcal{O}(e_2) - \dim \mathcal{O}(e_1) = \dim \mathfrak{p}^{e_1} - \dim \mathfrak{p}^{e_2}$.*

(b) Si il y a égalité et e_1 est presque $\mathfrak{p}$-distingué, alors $\mathfrak{C}(e_1) \subset \mathfrak{C}(e_2)$.

Démonstration. (a) Rappelons que l'on a une application de projection $\mathrm{pr}_1 : \mathfrak{C}(e_2) \to \overline{\mathcal{O}(e_2)}$. Comme $(g.e_2, g.e_2) \in \mathfrak{C}(e_2)$ pour tout $g \in K$ et comme $e_1 \in \overline{\mathcal{O}(e_2)}$, on a $(e_1, e_1) \in \mathfrak{C}(e_2)$ et e_1 appartient à l'image de pr_1. Maintenant, pour tout $x \in \mathcal{O}(e_2)$, on a $\mathrm{pr}_1^{-1}(x) = (x, \mathfrak{p}^x \cap \mathcal{N})$. On en déduit donc que $\mathrm{pr}_1^{-1}(e_1) \subseteq (e_1, \mathfrak{p}^{e_1} \cap \mathcal{N})$ est un sous ensemble de dimension au moins égale à $\dim \mathfrak{p}^{e_2} \cap \mathcal{N}$. On en déduit que $\dim \mathfrak{p}^{e_2} - \delta(e_2) = \dim \mathfrak{p}^{e_2} \cap \mathcal{N} \leqslant \dim \mathfrak{p}^{e_1} \cap \mathcal{N} = \dim \mathfrak{p}^{e_1} - \delta(e_1)$. L'inégalité de (a) s'en suit facilement.

(b) Si les hypothèses de (b) sont vérifiées, alors $\mathfrak{p}^{e_1} \cap \mathcal{N} = \bigoplus_{i \geqslant 1} \mathfrak{p}(e_1, i)$ est un ensemble irréductible contenant une fibre de pr_1 de même dimension, d'où $(e_1, \mathfrak{p}^{e_1} \cap \mathcal{N}) \subset \mathfrak{C}(e_2)$ et $\mathfrak{C}(e_1) \subset \mathfrak{C}(e_2)$. □

Définition 4.0.1. Si e_1 est presque $\mathfrak{p}$-distingué et e_2 nilpotent tel que $\mathcal{O}(e_1) \subset \overline{\mathcal{O}(e_2)}$ et $\delta(e_1) - \delta(e_2) = \dim \mathfrak{p}^{e_1} - \dim \mathfrak{p}^{e_2}$, on dit que $\mathcal{O}(e_1)$ se réduit en $\mathcal{O}(e_2)$ ou encore que $\mathcal{O}(e_2)$ est une réduction d'ordre $\delta(e_1) - \delta(e_2)$ de $\mathcal{O}(e_1)$.

Une telle réduction est dite minimale si il n'existe pas d'orbite $\mathcal{O}$ telle que $\overline{\mathcal{O}(e_1)} \subsetneq \overline{\mathcal{O}} \subsetneq \overline{\mathcal{O}(e_2)}$.

Lemme 4.0.2. *Si une orbite $\mathcal{O}(e_1)$ se réduit en une orbite $\mathcal{O}(e_2)$ alors toute orbite $\mathcal{O}(e_3)$ telle que $\overline{\mathcal{O}(e_1)} \subsetneq \overline{\mathcal{O}(e_3)} \subsetneq \overline{\mathcal{O}(e_2)}$, est une réduction de $\mathcal{O}(e_1)$.*

Démonstration. Par la proposition 4.0.1 (a), on a

$$\begin{aligned} \delta(e_1) - \delta(e_3) &= (\delta(e_1) - \delta(e_2)) - (\delta(e_3) - \delta(e_2)) \\ &\geqslant \big(\dim \mathfrak{p}^{e_1} - \dim \mathfrak{p}^{e_2}\big) - \big(\dim \mathfrak{p}^{e_3} - \dim \mathfrak{p}^{e_2}\big) \\ &\geqslant \dim \mathfrak{p}^{e_1} - \dim \mathfrak{p}^{e_3}. \end{aligned}$$

L'inégalité inverse découle de 4.0.1 (a). □

En vertu de la proposition et du lemme précédents, pour montrer à l'aide d'une réduction qu'un élément e_1 presque $\mathfrak{p}$-distingué n'engendre pas de composante irréductible étrange, il suffit de savoir qu'il existe une réduction minimale de e_1.

Dans chaque cas classique simple non encore résolu, on introduit un ordre sur les (ab-)diagrammes pour transférer la relation d'inclusion des orbites sur les diagrammes (cf. [Oh2, 1.9], [Oh1, 1.4]). C'est notamment l'objet des définitions suivantes.

Définition 4.0.3. Soit Γ un (ab-)diagramme de Young. On appelle motif de Γ un sous-diagramme obtenu à partir de Γ en conservant uniquement les lignes de longueur $d \in I$ où I est un intervalle de $\mathbb{N}$.

On note Γ' le diagramme obtenu en enlevant la première colonne de Γ. On définit par récurrence $\Gamma^{(k)} = (\Gamma^{(k-1)})'$ pour $k \in \mathbb{N}^*$ où $\Gamma^{(0)} = \Gamma$.

Soit (Γ_1, Γ_2) un couple de (ab-)diagrammes de Young. On définit une relation d'ordre partiel comme suit. On dit que $\Gamma_1 \leqslant \Gamma_2$ ou que Γ_2 est une dégénérescence de Γ_1 si :

- pour tout $k \in \mathbb{N}^*$, le nombre de cases de $\Gamma_1^{(k)}$ est inférieur à celui de $\Gamma_2^{(k)}$, lorsque Γ_1, Γ_2 sont des diagrammes de Young ;
- pour tout $k \in \mathbb{N}^*$, le nombre de a et le nombre de b de $\Gamma_1^{(k)}$ sont inférieurs respectivements à ceux de $\Gamma_2^{(k)}$, lorsque Γ_1, Γ_2 sont des ab-diagrammes de Young.

La dégénérescence $\Gamma_1 < \Gamma_2$ est dite minimale s'il n'existe pas d'(ab-)diagramme Γ tel que $\Gamma_1 < \Gamma < \Gamma_2$.

On appelle ligne commune à Γ_1 et Γ_2

- une ligne de même longueur si Γ_1 et Γ_2 sont des diagrammes de Young ;

- une ligne de même longueur débutant par la même lettre si Γ_1, Γ_2 sont des ab-diagrammes de Young.

On définit $\overline{\Gamma_1}$ et $\overline{\Gamma_2}$, en enlevant toutes les lignes communes à Γ_1 et Γ_2.

Rappelons que si $\mathcal{O}_1, \mathcal{O}_2$ sont deux orbites ayant pour (ab-)diagrammes respectifs Γ_1, Γ_2, alors $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$ implique $\Gamma_1 \leqslant \Gamma_2$. Réciproquement, si deux (ab-)diagrammes vérifient $\Gamma_1 \leqslant \Gamma_2$, et si $\mathcal{O}_1$ a pour diagramme Γ_1, alors il existe $\mathcal{O}_2$ correspondant à Γ_2 telle que $\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$ (cf. [Oh2, Théorème 3], [Oh1, Théorème 1]).

Notons que le fait qu'une orbite se réduise en une autre ne fait intervenir que des invariants propres aux (ab-)diagrammes de Young : défaut, dimension de centralisateur, inclusion. On transfert alors également les notions de défaut et de dimension de centralisateur aux (ab-)diagrammes que l'on note repectivement $\delta(\Gamma)$ et $\dim \mathfrak{p}^{\Gamma}$.

Définition 4.0.4. Soient Γ_1, Γ_2 deux (ab-)diagrammes de Young, on pose $\Delta = \Delta(\Gamma_1, \Gamma_2) = \delta(\Gamma_1) - \delta(\Gamma_2)$ et $s = s(\Gamma_1, \Gamma_2) = \dim \mathfrak{p}^{\Gamma_1} - \dim \mathfrak{p}^{\Gamma_2}$.

On dit que Γ_1 se réduit en Γ_2, ou que Γ_2 est une réduction de Γ_1, si $\Gamma_1 < \Gamma_2$ et $s = \Delta$.

Remarque 4.0.5. Les notions de défaut, de dimension de centralisateur et de réduction pour les (ab-)diagrammes dépendent du cas simple considéré (AI, AII, BDI ou CI).

Soit $\mathcal{O}_1$ une orbite et Γ_1 son (ab-)diagramme. L'existence d'une réduction pour $\mathcal{O}_1$ équivaut à celle d'une réduction pour Γ_1.

La notion de dégénérescence minimale équivaut à celle d'« adjacent degeneration » introduite dans [Oh1, (1.4)] et [Oh2, (2.4)].

4.1 Les cas AI et AII

Regardons le cas le plus simple : le cas AI. On fixe un élément e_1 presque $\mathfrak{p}$-distingué, et on note son diagramme de Young Γ_1. Par la proposition 3.1.1, les lignes de Γ_1 sont de longueurs distinctes. D'après [Oh1, Lemme 5], la seule dégénéréscence minimale Γ_2 de Γ_1 susceptible d'être une réduction doit fournir

C'est à dire que si l'on note $\Gamma_i = (p_1^i, p_2^i, \dots, p_n^i)$ où $(p_j^i)_j$ est une suite décroissante d'éléments de somme n, alors il existe $j_0 \in [\![1, n]\!]$ tel que

$$p_{j_0}^2 = p_{j_0}^1 + 1, \qquad p_{j_0+1}^2 = p_{j_0+1}^1 - 1, \qquad p_j^2 = p_j^1 \text{ sinon.}$$

La dimension du centralisateur dans $\mathfrak{p}$ d'un élément correspondant à un tel diagramme de Young est donnée par (cf. [Se1, 3.1]) :

$$\dim \mathfrak{p}^{\Gamma_i} = \Big(\sum_{j \in \mathbb{N}^*} j p_j^i\Big) - 1. \tag{4.1}$$

On peut donc calculer la différence s :

$$s = \dim \mathfrak{p}^{\Gamma_1} - \dim \mathfrak{p}^{\Gamma_2} = j_0 p^1_{j_0} + (j_0+1)p^1_{j_0+1} - j_0(p^1_{j_0}+1) - (j_0+1)(p^1_{j_0+1}-1) = 1.$$

Pour obtenir une réduction, on doit donc avoir $\Delta = 1$, c'est à dire $\delta(\Gamma_1) = \delta(\Gamma_2) + 1$ ce qui se traduit par le fait que Γ_2 possède une ligne de moins que Γ_1 (cf. proposition 3.1.1). Ceci n'est possible que si Γ_1 possède une ligne (nécessairement unique) de longueur 1. Ainsi, si l'on reprend les diagrammes de l'exemple 3.1.1, on voit que Γ_1 et Γ_1' possèdent chacun une réduction, mais que ce n'est le cas ni pour Γ_2, ni pour Γ_3. Au passage, ceci permet de prouver la conjecture 1.4.2 dans le cas AI de rang 2 et 3, car dans ces cas Γ_1 et Γ_1' représentent les seuls éléments presque $\mathfrak{p}$-distingués non $\mathfrak{p}$-distingués.

Dans le cas AII, par la même méthode, nous pouvons montrer que la seule dégénérescence minimale possible pour un élément presque $\mathfrak{p}$-distingué est obtenue de celle du cas AI en doublant les lignes. On a alors $s = 4$ tandis que $\Delta = 1$, on n'a donc pas de réduction d'élément presque $\mathfrak{p}$-distingué.

Conséquence. La conjecture 1.4.2 est démontrée dans le cas AI jusqu'au rang 3 et dans le cas AII de rang 1.

4.2 Les cas BDI et CI

Heureusement, dans les cas BDI et CI, la méthode de réduction permet d'obtenir plus de résultats. On traite tout d'abord le cas BDI, le cas CI lui étant très similaire.

Introduisons quelques notions pour calculer les entiers $s(\Gamma_1, \Gamma_2)$. On fixe un élément e nilpotent quelconque (non nul) dans une algèbre symétrique de type BDI, et Γ désigne son *ab*-diagramme de Young.

Définition 4.2.1. Si Γ ne comporte que des lignes de longueur impaire on définit k_j^Γ comme étant la longueur de la $(2j+1)$-ème colonne de Γ.

On rappelle que le défaut de la longueur d est donné par $\delta^\Gamma(d) = \min(a_d, b_d)$ et que $\delta(\Gamma) = \sum_d \delta^\Gamma(d)$ (cf. section 3.4 et remarque 3.0.3).

Lemme 4.2.2. *Si h est un élément à valeurs propres entières de $\mathfrak{so}(V)$ et $V(i)$ le h-module de poids $i \in \mathbb{Z}$, alors $\mathfrak{g}^h \cong \mathfrak{so}(V(0)) \oplus \bigoplus_{i>0} \mathfrak{gl}(V(i))$.*

Démonstration. On note Φ la forme bilinéaire symétrique canonique définissant $\mathfrak{so}(V)$. Soit $v \in V(i)$ et $w \in V(j)$. On a $\Phi(h.v, w) + \Phi(v, h.w) = 0$, d'où $\Phi(v, w) = 0$ ou $i + j = 0$. On a donc une décomposition orthogonale h-invariante $V = V(0) \oplus \bigoplus_{i \in \mathbb{N}^*}(V(i) \oplus V(-i))$. On voit que $(\mathfrak{g}^h)_{|V(i)\oplus V(-i)} \cong \mathfrak{gl}(V(i))$. D'où le résultat. □

Lemme 4.2.3. *Si Γ ne comporte que des lignes de longueur impaire, on a*

$$\dim \mathfrak{p}^{\Gamma} = C + \frac{k_0^{\Gamma}(k_0^{\Gamma}-1)}{4} + \frac{1}{2}\sum_{j>0}(k_j^{\Gamma})^2$$

où C est une constante qui ne dépend que de l'algèbre symétrique considérée.

Démonstration. On inclut e dans un $\mathfrak{sl}_2$-triplet normal (e,h,f). L'élément e est pair donc $\dim \mathfrak{p}^e = (\dim \mathfrak{p} - \frac{1}{2}\dim \mathfrak{g}) + \frac{1}{2}\dim \mathfrak{g}^h$. Comme Γ ne contient que des lignes de longueur impaire, on peut décomposer V en $\bigoplus_{j\in\mathbb{Z}} V(2j)$. On voit alors par la théorie des représentations de $\mathfrak{sl}_2$ que $k_j^{\Gamma} = \dim V(2j)$ pour $j \in \mathbb{N}$. On obtient ensuite par le lemme 4.2.2 que $\frac{1}{2}\dim \mathfrak{g}^h = \frac{k_0^{\Gamma}(k_0^{\Gamma}-1)}{4} + \frac{1}{2}\sum_{j>0}(k_j^{\Gamma})^2$. □

On fixe maintenant un élément presque $\mathfrak{p}$-distingué e_1 et on note Γ_1 son ab-diagramme de Young. On énumère les dégénérescences minimales possibles (cf. [Oh2, Table V cas BDI]). La première colonne désigne le numéro de la dégénérescence minimale $\Gamma_1 < \Gamma_2$ telle qu'elle est indiquée dans [Oh2]. Les deux colonnes suivantes donnent $\overline{\Gamma_1}$ et $\overline{\Gamma_2}$. La quatrième donne l'entier $s = \dim \mathfrak{p}^{\Gamma_1} - \dim \mathfrak{p}^{\Gamma_2}$ et la dernière indique les restrictions sur les entiers p et q.

Notons que le cas (1) de [Oh2] n'apparaît pas car Γ_1 ne peut alors pas correspondre à une orbite presque $\mathfrak{p}$-distinguée. Les cas (6) et (7) n'apparaissent pas car, pour ces dégénérescences, on a $\Delta \leqslant 0$. L'entier s a été calculé à partir du lemme 4.2.3, sauf dans certains cas dont on verra plus tard qu'ils ne peuvent donner lieu à de nouvelles réductions.

numéro de [Oh2]	$\overline{\Gamma_1}$	$\overline{\Gamma_2}$	s	précisions
(2)	$\overbrace{ab.....ba}^{2p+1}$ $\underbrace{ab...ba}_{2q+1}$	$\overbrace{ab.....ba}^{2p+3}$ $\underbrace{ab...ba}_{2q-1}$	$k_q^{\Gamma_1} - k_{p+1}^{\Gamma_1} - 1$	$p \geqslant q \geqslant 1$
(3)	$\overbrace{ab.....ba}^{2p+1}$ $\underbrace{ba...ab}_{2q+1}$	$\overbrace{ab.....ba}^{2p+3}$ $\underbrace{ba...ab}_{2q-1}$	$k_q^{\Gamma_1} - k_{p+1}^{\Gamma_1} - 1$	$p \geqslant q \geqslant 1$
(4)	$\overbrace{ab.....ba}^{2p+1}$ $ba.....ab$ $\underbrace{ab...ba}_{2q+1}$	$\overbrace{ab.....ab}^{2p+2}$ $ba.....ba$ $\underbrace{ab...ba}_{2q-1}$	- - - -	$p \geqslant q \geqslant 1$
(5)	$\overbrace{ab.....ba}^{2p+1}$ $ab...ba$ $\underbrace{ba...ab}_{2q+1}$	$\overbrace{ab.....ba}^{2p+3}$ $ab...ab$ $\underbrace{ba...ba}_{2q}$	$k_0^{\Gamma_1} - k_{p+1}^{\Gamma_1} - 2$ - - - -	$p \geqslant q = 0$ $p \geqslant q \geqslant 1$
(8)	$\overbrace{ba.....ab}^{2p+1}$ $\underbrace{ab...ba}_{2q+1}$	$\overbrace{ab.....ba}^{2p+3}$ $\underbrace{ba...ab}_{2q-1}$	$k_q^{\Gamma_1} - k_{p+1}^{\Gamma_1} - 1$	$p \geqslant q \geqslant 1$
(9)	$\overbrace{ba.....ab}^{2p+1}$ $ab...ba$ $\underbrace{ab...ba}_{2q+1}$	$\overbrace{ba.....ab}^{2p+3}$ $ba...ba$ $\underbrace{ab...ab}_{2q}$	$k_0^{\Gamma_1} - k_{p+1}^{\Gamma_1} - 2$ - - - -	$p \geqslant q = 0$ $p \geqslant q \geqslant 1$
(10)	$\overbrace{ba.....ab}^{2p+1}$ $ba.....ab$ $\underbrace{ab...ba}_{2q+1}$	$\overbrace{ba.....ba}^{2p+2}$ $ab.....ab$ $\underbrace{ab...ba}_{2q-1}$	- - - -	$p \geqslant q \geqslant 1$

Pour une dégénérescence minimale donnée, on note $\Delta(d) = \delta^{\Gamma_1}(d) - \delta^{\Gamma_2}(d)$. Notons que $\Delta(d) \in \{0, 1, -1\}$ par le tableau de la section 3.4.

Considérons la dégénérescence minimale (3). D'après le lemme 4.2.3, on a

$$s = k_q - k_{p+1} - 1. \tag{4.2}$$

On voit que les réductions éventuelles de défaut ne peuvent se produire que pour les longueurs $2q + 1$ et $2p + 1$. Les augmentations éventuelles de défaut, pour les longueurs $2p + 3$ et $2q - 1$.

On va tout d'abord se placer dans le cas $p = q$. On a alors $a_{2p+1}b_{2p+1} \neq 0$, or e_1 est presque distingué donc $a_{2p+1}b_{2p+1} = 1$. On en déduit $k_q = k_p = k_{p+1} + 2$ et $s = 1$ d'après (4.2). Or $\Delta(2p+1) = 1$. Pour obtenir une réduction, on doit nécessairement avoir $\Delta(2p+3) = \Delta(2p-1) = 0$. Voyons ce que cela donne dans les différents cas :

- Si $a_{2p+3}b_{2p+3} = 1$ ou $b_{2p+3} = 0$ (resp. $a_{2p-1}b_{2p-1} = 1$ ou $a_{2p-1} = 0$), alors $\Delta(2p + 3) = 0$ (resp. $\Delta(2p - 1) = 0$).
- Le seul cas pour lequel (3) ne donne pas de réduction est celui où $b_{2p+3} = m_{2p+3} \neq 0$ ou $a_{2p-1} = m_{2p-1} \neq 0$.
- On obtient un résultat analogue si l'on considère la réduction (3') où les rôles de a et b sont inversés. On voit alors que le seul cas qui ne peut pas être réduit par (3) ou (3') est (à permutation de a et b près) le cas où $a_{2p+3} = m_{2p+3} \neq 0$ et $a_{2p-1} = m_{2p-1} \neq 0$. Le plus petit exemple est le suivant :

$$\begin{array}{l} ababa \\ aba \\ bab \\ a \end{array}.$$

Notons que l'on a ainsi montré que si des longueurs de lignes consécutives impaires avaient un défaut, on pourrait réduire l'orbite $\mathcal{O}(e_1)$. Nous supposerons maintenant dans la suite que Γ_1 ne possède pas de motif

$$\begin{array}{l} \overbrace{ab.....ba}^{2p+3} \\ ba.....ab \\ ab...ba \\ \underbrace{ba...ab}_{2p+1} \end{array}.$$

Nous allons aussi montrer que (3) pour $p \neq q$ ne peut pas apporter de réduction. En effet, soit Γ_2 le diagramme obtenu grâce à une dégénérescence (3) où l'on suppose $p \neq q$. Avec les notations précédentes, il est facile de voir que si $\Delta(2p + 1) = 1$ (resp. $\Delta(2q + 1) = 1$) alors $k_{p+1} = k_p - 2$ (resp. $k_q = k_p + 2$). On en déduit alors que

$$k_q \geqslant k_p + 1 + \Delta(2q + 1), \qquad k_{p+1} \leqslant k_p - 1 - \Delta(2p + 1),$$

$$s = k_q - k_{p+1} - 1 \geqslant (1 + \Delta(2q + 1)) + (1 + \Delta(2p + 1)) - 1 \geqslant \Delta + 1.$$

On ne peut donc pas espérer de réduction. Le cas de l'opération (2) se traite de la même manière.

Regardons maintenant ce qui se passe dans le cas de la dégénérescence minimale (5) pour $q = 0$ et p quelconque. On remarque que $a_1b_1 = 1$ car e_1 est presque $\mathfrak{p}$-distingué. D'après le lemme 4.2.3, on a $s = k_0 - k_{p+1} - 2$. Observons que $k_0 = k_p + 2$ et $\Delta(1) = 1$. Notons que le cas où $\delta^{\Gamma_1}(2p+1) = 1$ a déjà été montré comme réductible (suivant que $p = 1$ auquel cas on a deux défauts consécutifs, ou $p > 1$ auquel cas $m_{2p-1} = 0$). On supposera donc que $\delta^{\Gamma_1}(2p+1) = 0$ et on obtient une réduction par (5) si et seulement si $k_{p+1} = k_p - 1$ c'est à dire $m_{2p+1} = 1$. Le seul cas que l'on n'arrive pas à traiter est le cas où $m_{2p+1} > 1$ dont le plus petit exemple est :

$$\begin{array}{l} aba \\ aba \\ a \\ b \end{array}.$$

On peut montrer que les autres dégénérescences minimales ne mènent pas à des réductions des exemples que l'on a cité. Donnons ici des exemples de réduction. On considère les diagrammes Γ_5 et Γ_6 de l'exemple 3.3.1 et on indique de l'autre côté de la flèche un diagramme qui les réduit.

$$\begin{array}{l} aba \\ a \\ b \end{array} \rightarrow ababa; \qquad \begin{array}{l} ababa \\ aba \\ bab \\ b \end{array} \rightarrow \begin{array}{l} ababa \\ ababa \\ b \\ b \end{array}.$$

En conclusion pour le cas BDI : on peut réduire e_1 si et seulement si les lignes de son ab-diagramme comportant un défaut ne sont pas toutes dans des motifs du type (à permutation de a et b près) :

$$\underbrace{\underbrace{\begin{array}{l} \overbrace{\qquad\qquad}^{2p+5} \\ \vdots \\ ab.........ba \\ ab......ba \\ ba......ab \\ ab...ba \\ \vdots \end{array}}_{2p+1}}_{2p+3} \quad \text{ou} \quad \begin{array}{l} \overbrace{\qquad}^{2p+3} \\ \vdots \\ ab...ba \\ ab...ba \\ a \\ b \end{array} \quad \text{pour } p \geqslant 0.$$

Dans le cas CI, les dégénérescences minimales possibles sont indexées dans [Oh2] de (1) à (10) de façon similaire au cas BDI. La dégénérescence (5) ne donne plus de réduction, seule, la (3) en donne une. Il existe un unique motif non réductible (à permutation de a et b près) :

$$\begin{array}{l} \overbrace{\qquad\qquad\quad}^{2p+6} \\ \vdots \\ ab.........ab \\ ab......ab \\ ba......ba \\ ab...ab \\ \vdots \\ \underbrace{\qquad\quad}_{2p+2} \\ \underbrace{\qquad\qquad}_{2p+4} \end{array} \qquad \text{pour } p \geqslant 0.$$

Indiquons, comme ci-dessus, comment les diagrammes Γ_8 et Γ_9 de l'exemple 3.3.1 se réduisent :

$$\begin{array}{l} baba \\ ba \\ ab \end{array} \longrightarrow \begin{array}{l} bababa \\ ab \end{array} ; \qquad \begin{array}{l} bababa \\ baba \\ abab \\ ab \end{array} \longrightarrow \begin{array}{l} bababa \\ bababa \\ ab \\ ab \end{array} .$$

Conséquence. La conjecture 1.4.2 est vraie pour $(\mathfrak{so}_n, \mathfrak{so}_{n_a} \times \mathfrak{so}_{n_b})$ (BDI) avec $n_a \leqslant 2$ ou $n_b \leqslant 2$ ou $max(n_a, n_b) \leqslant 4$. Dans le cas (CI), elle est vraie jusqu'au rang 7 ; *i.e.* pour $(\mathfrak{sp}_{2n}, \mathfrak{gl}_n)$ avec $n \leqslant 7$.

5 Une utilisation complète du lemme 1.4.2

5.1 Étude de $\mathfrak{g}(e, 1)$

Nous avons jusqu'à présent écarté le cas où $\mathfrak{p}(e, 0)$ contient des éléments nilpotents grâce à la proposition 1.4.1. L'objet de cette section est d'utiliser le lemme 1.4.2, lorsque $\mathfrak{p}(e, 1)$ est non nul dans les cas classiques qui nous restent. Notons que dans les cas BDI et CII, les presque $\mathfrak{p}$-distingués sont pairs. En particulier dans ces deux cas, pour un élément presque $\mathfrak{p}$-distingué e, on a $\mathfrak{p}(e, 1) = \{0\}$. D'après le commentaire qui suit le lemme 1.4.3, le lemme 1.4.2 est impuissant pour éliminer la possibilité qu'un tel e engendre une composante étrange. C'est pourquoi nous allons nous limiter aux cas AI et AII. Dans cette section on fixe donc $\mathfrak{g} = \mathfrak{sl}(V)$ et un élément nilpotent non nul e que l'on inclut dans un $\mathfrak{sl}_2$-triplet (e, h, f). Comme dans [Oh1], on note $(\lambda_i)_{i \in \mathbb{N}}$, la partition de n correspondant au diagramme de Young de e. Alors, on sait qu'il existe une base $\{e^a.v_i \mid a \leqslant \lambda_i - 1; a, i \in \mathbb{N}, v_i \in V\}$ de V (cf. [TY] 19.2) pour laquelle h est diagonal. Plus précisément,

$$V_i := \langle e^a.v_i \mid a \leqslant \lambda_i - 1 \rangle$$

est un $\mathfrak{sl}_2$-module irréductible et en notant $V(k)$ le h-espace propre de V associé à k, on a

$$V(k) = \langle \{e^a.v_i \mid 2a - \lambda_i + 1 = k\} \rangle. \tag{5.1}$$

Le lemme suivant nous indique des cas où il existe un élément $e_1 \in \mathfrak{g}^e$ tel que $e_1 \notin \overline{\mathcal{O}(e)}$.

Lemme 5.1.1. *Supposons qu'il existe $i_1, i_2 \in \mathbb{N}$ tels que $0 < \lambda_{i_1} = \lambda_{i_2} - 1$. Alors, il existe un élément $e_1 \in \bigoplus_{i \geqslant 1} \mathfrak{g}(e,i)$ tel que $e_1 \notin \overline{\mathcal{O}(e)}$.*

Démonstration. On note $e = \sum_{i \in \mathbb{N}} e_i$ où e_i est défini par $\begin{cases} e \text{ sur } V_i \\ 0 \text{ sur } \bigoplus_{j \neq i} V_j. \end{cases}$ Les éléments e_i commutent deux à deux. Définissons maintenant l'élément

$$e_{i_1,i_2} : \begin{cases} e^a.v_{i_1} \to e^{a+1}.v_{i_2} \text{ pour } 0 \leqslant a \leqslant \lambda_{i_1} - 1 \\ e^a.v_{i_2} \to e^a.v_{i_1} \text{ pour } 0 \leqslant a \leqslant \lambda_{i_1} - 1. \\ e^a.v_j \to 0 \text{ sinon.} \end{cases} \tag{5.2}$$

On vérifie facilement que $e_{i_1} + e_{i_2} = e_{i_1,i_2}^2$, et donc que e commute avec $e_1 := e_{i_1,i_2} + \sum_{i \neq i_1,i_2} e_i$. Par ailleurs le diagramme de e_1 est strictement supérieur à celui de e, donc $e_1 \notin \overline{\mathcal{O}(e)}$. Enfin, l'égalité (5.1) nous donne $e_{i_1,i_2}.(V(k)) \subseteq V(k+1)$ donc $e_1 \in \mathfrak{g}(e,1) \oplus \mathfrak{g}(e,2)$. $\square$

D'après le lemme 1.4.3, on a $\bigoplus_{i \geqslant 2} \mathfrak{g}(e,i) \subset \overline{\mathcal{O}(e)}$. Le lemme suivant peut alors être considéré comme une réciproque du lemme 5.1.1.

Proposition 5.1.1. *Supposons que pour tous $i, j \in \mathbb{N}$ tels que $\lambda_i, \lambda_j \neq 0$, on ait $\lambda_i - \lambda_j \neq 1$. Alors $\mathfrak{g}(e,1) = \{0\}$.*

Démonstration. Nous allons montrer la contraposée en supposant l'existence de $x \in \mathfrak{g}(e,1) \setminus \{0\}$. Commençons par noter que d'après (5.1), on a pour tous $j \in \mathbb{N}^*$, $k \in \mathbb{N}$:

$$v_j \in V(-\lambda_j + 1) \text{ et } \ker(e^k) \cap V(-k+1) = \langle v_i \mid \lambda_i = k \rangle. \tag{5.3}$$

Comme x commute avec e, il existe un indice i tel que $y = x.v_i \neq 0$ et puisque $x \in \mathfrak{g}(1,h)$, on a $y \in V(-\lambda_i + 2) = e.V(-\lambda_i) \oplus \ker(e^{\lambda_i - 1})$. Soit $y_1 + y_2$ la décomposition de y dans cette somme directe. Soit $z \in V(-\lambda_i)$ tel que $y_1 = e.z$. On a $e^{\lambda_i+1}.z = x.(e^{\lambda_i}.v_i) = 0$ donc $z \in \langle v_j \mid \lambda_j = \lambda_i + 1 \rangle$. Si $z \neq 0$, il existe nécessairement j tel que $\lambda_j = \lambda_i + 1$. Dans le cas contraire, $y_1 = e.z = 0$ et $0 \neq y = y_2 \in \ker(e^{\lambda_i - 1}) \cap V(-\lambda_i + 2) = \langle v_j \mid \lambda_j = \lambda_i - 1 \rangle$. D'où l'existence de j tel que $\lambda_j = \lambda_i - 1$. $\square$

Grâce au lemme 1.4.3, on voit que si e est presque $\mathfrak{p}$-distingué et vérifie les hypothèses de la proposition précédente alors $\mathcal{N} \cap \mathfrak{p}^e \subset \overline{\mathcal{O}(e)}$. Le lemme 1.4.2 ne peut donc rien nous apporter. Dans le cas contraire nous allons pouvoir utiliser ce lemme ; c'est l'objet de la suite de cette section.

5.2 Le cas AI

On utilise les notations de la sous-section précédente. Soit e presque $\mathfrak{p}$-distingué ; les λ_i sont donc deux à deux distincts. De plus, prenant en considération la proposition 5.1.1, nous ferons l'hypothèse qu'il existe deux indices i_1 et i_2 tels que $0 < \lambda_{i_1} = \lambda_{i_2} - 1$. On sait qu'on peut alors supposer que la forme bilinéaire symétrique sur V définissant la paire symétrique $(\mathfrak{g}, \mathfrak{k})$ est donnée par (cf. [Oh1, Lemme 1])

$$(e^a.v_i, e^b.v_j) = \begin{cases} 1 \text{ si } i = j \text{ et } a+b+1 = \lambda_i \\ 0 \text{ sinon.} \end{cases}$$

Il est facile de vérifier que pour tous a, b, i, j, on a $(e_1.e^a.v_i, e^b.v_j) = (e^a.v_i, e_1.e^b.v_j)$ où $e_1 = e_{i_1,i_2} + \sum_{j \neq i_1,i_2} e_i$ est l'élément de la démonstration du lemme 5.1.1. Ceci implique que $e_1 \in \mathfrak{p}$ et on a obtenu un élément de $\mathfrak{p}^e$ qui n'appartient pas à $\overline{\mathcal{O}(e)}$. D'après le lemme 1.4.2, la sous-variété $\mathfrak{C}(e)$ n'est donc pas une composante irréductible de $\mathfrak{C}^{\text{nil}}(\mathfrak{p})$.

Concrètement, cela veut par exemple dire que l'élément correspondant au diagramme de Young Γ_2 de l'exemple 3.1.1 ne peut pas engendrer de composante irréductible étrange.

Conséquence. La conjecture 1.4.2 est démontrée dans le cas AI de rang 4.

5.3 Le cas AII

Nous allons cette fois nous placer dans le cas AII. Soit e presque $\mathfrak{p}$-distingué ; donc si $\lambda_i \neq 0$, il existe un unique indice $\beta(i)$ tel que $\lambda_{\beta(i)} = \lambda_i$. De plus, cf. proposition 5.1.1, nous allons supposer qu'il existe i_1 et i_2 tels que $0 < \lambda_{i_1} = \lambda_{i_2} - 1$. On peut alors supposer que la forme bilinéaire antisymétrique sur V définissant $(\mathfrak{g}, \mathfrak{k})$ est donnée par (cf. [Oh1, Lemme 1])

$$(e^a.v_i, e^b.v_j) = \begin{cases} \alpha(i) \text{ si } i = \beta(j) \text{ et } a+b+1 = \lambda_i \\ 0 \text{ sinon.} \end{cases}$$

où $\alpha(i) \in \{\pm 1\}$ et $\alpha(\beta(i)) = -\alpha(i)$. Quitte à permuter i_1 et $\beta(i_1)$, on peut supposer $\alpha(i_1) = \alpha(i_2)$. Avec des notations de la démonstration du lemme 5.1.1 et en particulier (5.2), on pose $e'_1 = e_{i_1,i_2} + e_{\beta(i_1),\beta(i_2)} + \sum_{i \notin \{i_1,i_2,\beta(i_1),\beta(i_2)\}} e_i$. Comme dans le cas AI, on vérifie que c'est un élément de $\mathfrak{p}^e$ qui n'appartient pas à $\overline{\mathcal{O}(e)}$. D'après le lemme 1.4.2, la sous-variété $\mathfrak{C}(e)$ n'est donc pas une composante irréductible de $\mathfrak{C}^{\text{nil}}(\mathfrak{p})$.

Ce qui précède implique par exemple que l'élément correspondant au diagramme de Young Γ_4 de l'exemple 3.1.1 ne peut pas engendrer de composante irréductible étrange.

Conséquence. La conjecture 1.4.2 est démontrée dans le cas AII de rang 3.

6 Les cas exceptionnels

Dans les cas exceptionnels, nous disposons d'une classification des orbites d'éléments nilpotents, des centralisateurs des $\mathfrak{sl}_2$-triplets associés, et de leurs relations d'inclusion, cf. [Dj1], [Dj2], [Dj5], [Dj6]. Nous allons donc pouvoir appliquer au cas exceptionnel les mêmes méthodes que précédemment visant à éliminer des possibilités, pour des orbites nilpotentes, d'engendrer des composantes étranges. Finalement, nous démontrerons la conjecture dans tout les cas exceptionnels hormis EI.

6.1 Utilisation de la proposition 1.4.1

Fixons une algèbre de Lie simple symétrique exceptionnelle $(\mathfrak{g}, \mathfrak{k})$ ainsi qu'une forme réelle du même type $\mathfrak{g}_{\mathbb{R}}$. Les tables de [Dj1] et [Dj2] nous donnent les orbites nilpotentes réelles dans $\mathfrak{g}_{\mathbb{R}}$. En fait, la classification de D.Z. Djokovic s'appuie sur une description des orbites complexes de $(\mathfrak{g}, \mathfrak{k})$ qu'il relie à la classification des orbites réelles par la correspondance de Kostant-Sekiguchi. Chaque ligne désigne donc une orbite complexe non nulle, que l'on repérera par son numéro donné dans la première colonne. Fixons un élément e de cette orbite, et incluons-le dans un $\mathfrak{sl}_2$-triplet (e, h, f) engendrant une algèbre de Lie $\mathfrak{s}$. Le centralisateur réductif réel donné dans la dernière colonne de ces tables est en réalité déduit du centralisateur complexe $\mathfrak{g}^{\mathfrak{s}}$ (cf. [Dj1, §15]). Tous les calculs de D.Z. Djokovic effectués dans le cas complexe restent vrais dans le cas d'un corps algébriquement clos de caractéristique zéro. Pour retrouver la paire réductive $(\mathfrak{g}^{\mathfrak{s}}, \mathfrak{k}^{\mathfrak{s}})$ complexe, il suffit d'appliquer la correspondance dans l'autre sens, sachant que V_r désigne un tore de dimension r dans $\mathfrak{p}$ et que T_r désigne un tore de dimension r dans $\mathfrak{k}$. On voit donc facilement si $\mathfrak{g}^{\mathfrak{s}} = \mathfrak{k} \oplus V_r$ ou non. Les tableaux suivants résultent de ces calculs dans les différents cas simples exceptionnels et donnent les orbites presque $\mathfrak{p}$-distingués. La première colonne donne le numéro de l'orbite tel qu'il apparaît dans [Dj1] et [Dj2]. La seconde colonne donne le type d'isomorphisme de la paire $(\mathfrak{g}^{\mathfrak{s}}, \mathfrak{k}^{\mathfrak{s}})$. La troisième donne le défaut d'un élément de cette orbite.

$E_{6(-26)}$ (cas EIV)

1	$(\mathfrak{so}_7 \oplus T_1, \mathfrak{so}_7)$	1
2	(G_2, G_2)	0

$F_{4(-20)}$ (cas FII)

1	$(\mathfrak{sl}_4, \mathfrak{sl}_4)$	0
2	(G_2, G_2)	0

$G_{2(2)}$ (cas GI)

3	$(0,0)$	0
4	$(0,0)$	0
5	$(0,0)$	0

$E_{6(-14)}$ (cas EIII)

3	$(\mathfrak{so}_7 \oplus T_1, \mathfrak{so}_7 \oplus T_1)$	0
4	$(\mathfrak{so}_7 \oplus T_1, \mathfrak{so}_7 \oplus T_1)$	0
7	$(\mathfrak{sl}_3 \oplus T_1, \mathfrak{sl}_3 \oplus T_1)$	0
8	$(\mathfrak{sl}_3 \oplus T_1, \mathfrak{sl}_3 \oplus T_1)$	0
9	(G_2, G_2)	0
10	$(\mathfrak{so}_5 \oplus T_1, \mathfrak{so}_5 \oplus T_1)$	0
11	$(\mathfrak{so}_5 \oplus T_1, \mathfrak{so}_5 \oplus T_1)$	0
12	$(\mathfrak{sl}_2 \oplus T_1, \mathfrak{sl}_5 \oplus T_1)$	0

$E_{6(6)}$ (cas EI)

12	(T_2, T_1)	1
16	$(T_1, 0)$	1
17	$(T_1, 0)$	1
18	$(0,0)$	0
19	$(0,0)$	0
20	$(0,0)$	0
21	$(T_1, 0)$	1
22	$(0,0)$	0
23	$(T_2, 0)$	2

$F_{4(4)}$ (cas FI)

6	$(\mathfrak{sl}_3, \mathfrak{sl}_3)$	0
16	$(0,0)$	0
17	$(0,0)$	0
18	$(0,0)$	0
19	$(\mathfrak{sl}_2, \mathfrak{sl}_2)$	0
22	$(0,0)$	0
23	$(0,0)$	0
24	$(0,0)$	0
25	$(0,0)$	0
26	$(0,0)$	0

$E_{7(7)}$ (cas EV)

16	(G_2, G_2)	0
17	(G_2, G_2)	0
39	$(\mathfrak{sl}_2, \mathfrak{sl}_2)$	0
40	$(\mathfrak{sl}_2, \mathfrak{sl}_2)$	0
48	(T_2, T_2)	0
49	(T_2, T_2)	0
50	$(T_2, 0)$	2
55	$(\mathfrak{sl}_2, \mathfrak{sl}_2)$	0
56	$(\mathfrak{sl}_2, \mathfrak{sl}_2)$	0
67	$(0, 0)$	0
68	$(0, 0)$	0
69	$(0, 0)$	0
70	$(0, 0)$	0
76	$(0, 0)$	0
77	$(0, 0)$	0
78	$(0, 0)$	0
79	$(0, 0)$	0
80	(T_1, T_1)	0
81	$(T_1, 0)$	1
85	$(0, 0)$	0
86	$(0, 0)$	0
87	$(0, 0)$	0
88	$(0, 0)$	0
89	$(0, 0)$	0
90	$(0, 0)$	0
91	$(0, 0)$	0
92	$(0, 0)$	0
93	$(0, 0)$	0
94	$(0, 0)$	0

$E_{8(8)}$ (cas EVIII)

14	(G_2, G_2)	0
15	(G_2, G_2)	0
34	$(\mathfrak{sl}_3, \mathfrak{sl}_3)$	0
42	$(\mathfrak{sl}_2 \oplus T_1, \mathfrak{sl}_2 \oplus T_1)$	0
45	$(2\mathfrak{sl}_2, 2\mathfrak{sl}_2)$	0
51	$(\mathfrak{sl}_3, \mathfrak{sl}_3)$	0
67	$(0, 0)$	0
68	$(0, 0)$	0
69	$(0, 0)$	0
70	$(2\mathfrak{sl}_2, 2\mathfrak{sl}_2)$	0
79	(T_1, T_1)	0
80	(T_1, T_1)	0
81	$(T_1, 0)$	1
84	(T_1, T_1)	0
85	$(T_1, 0)$	1
87	(T_1, T_1)	0
88	$(T_1, 0)$	1
91	$(0, 0)$	0
92	$(0, 0)$	0
93	(T_1, T_1)	0
94	(T_1, T_1)	0
95	$(T_1, 0)$	1
98	$(0, 0)$	0
99	$(0, 0)$	0
101	$(0, 0)$	0
102	$(0, 0)$	0
104	$(0, 0)$	0
105	$(0, 0)$	0
106	$(0, 0)$	0
107	$(0, 0)$	0
109	$(0, 0)$	0
110	$(0, 0)$	0
111	$(0, 0)$	0
112	$(0, 0)$	0
113	$(0, 0)$	0
114	$(0, 0)$	0
115	$(0, 0)$	0

$E_{6(2)}$ (cas EII)

6	$(2\mathfrak{sl}_2, 2\mathfrak{sl}_2)$	0
12	$(\mathfrak{sl}_2 \oplus T_1, \mathfrak{sl}_2 \oplus T_1)$	0
13	$(\mathfrak{sl}_2 \oplus T_1, \mathfrak{sl}_2 \oplus T_1)$	0
20	(T_2, T_2)	0
21	(T_2, T_2)	0
22	(T_2, T_1)	1
23	$(\mathfrak{sl}_3, \mathfrak{sl}_3)$	0
25	$(\mathfrak{sl}_2 \oplus T_1, \mathfrak{sl}_2 \oplus T_1)$	0
27	(T_1, T_1)	0
28	(T_1, T_1)	0
29	(T_1, T_1)	0
30	(T_1, T_1)	0
32	$(0, 0)$	0
33	$(0, 0)$	0
34	(T_1, T_1)	0
35	(T_1, T_1)	0
36	$(0, 0)$	0
37	$(0, 0)$	0

$E_{7(-5)}$ (cas EVI)

6	$(\mathfrak{sl}_6, \mathfrak{sl}_6)$	0
14	$(G_2 + \mathfrak{sl}_2, G_2 + \mathfrak{sl}_2)$	0
19	$(3\mathfrak{sl}_2, 3\mathfrak{sl}_2)$	0
20	$(3\mathfrak{sl}_2, 3\mathfrak{sl}_2)$	0
22	$(\mathfrak{sp}_6, \mathfrak{sp}_6)$	0
24	$(\mathfrak{sl}_2 \oplus T_1, \mathfrak{sl}_2 \oplus T_1)$	0
25	$(\mathfrak{sl}_3 \oplus T_1, \mathfrak{sl}_3 \oplus T_1)$	0
27	(T_2, T_2)	0
28	$(\mathfrak{sl}_2 \oplus T_1, \mathfrak{sl}_2 \oplus T_1)$	0
29	$(\mathfrak{sl}_2, \mathfrak{sl}_2)$	0
31	$(\mathfrak{sl}_2, \mathfrak{sl}_2)$	0
32	$(\mathfrak{sl}_2, \mathfrak{sl}_2)$	0
33	$(2\mathfrak{sl}_2, 2\mathfrak{sl}_2)$	0
34	$(2\mathfrak{sl}_2, 2\mathfrak{sl}_2)$	0
35	$(\mathfrak{sl}_2, \mathfrak{sl}_2)$	0
36	(T_1, T_1)	0
37	$(\mathfrak{sl}_2, \mathfrak{sl}_2)$	0

$E_{7(-25)}$ (cas EVII)

6	(F_4, F_4)	0
7	(F_4, F_4)	0
11	$(\mathfrak{sl}_4 \oplus T_1, \mathfrak{sl}_4 \oplus T_1,)$	0
12	$(\mathfrak{sl}_4 \oplus T_1, \mathfrak{sl}_4 \oplus T_1,)$	0
16	$(\mathfrak{so}_7, \mathfrak{so}_7)$	0
17	$(\mathfrak{so}_7, \mathfrak{so}_7)$	0
18	$(\mathfrak{so}_7, \mathfrak{so}_7)$	0
19	$(\mathfrak{so}_7, \mathfrak{so}_7)$	0
20	$(\mathfrak{sl}_3 \oplus T_1, \mathfrak{sl}_3 \oplus T_1)$	0
21	(G_2, G_2)	0
22	(G_2, G_2)	0

$E_{8(-24)}$ (cas EIX)

6	(E_6, E_6)	0
18	$(\mathfrak{so}_8, \mathfrak{so}_8)$	0
19	$(\mathfrak{so}_8, \mathfrak{so}_8)$	0
21	(F_4, F_4)	0
23	$(\mathfrak{so}_5 \oplus T_1, \mathfrak{so}_5 \oplus T_1)$	0
24	$(\mathfrak{sl}_5, \mathfrak{sl}_5)$	0
26	$(\mathfrak{sl}_3 \oplus T_1, \mathfrak{sl}_3 \oplus T_1)$	0
27	$(\mathfrak{sl}_4, \mathfrak{sl}_4)$	0
28	$(2\mathfrak{sl}_2, 2\mathfrak{sl}_2)$	0
30	(G_2, G_2)	0
31	(G_2, G_2)	0
32	$(\mathfrak{so}_7, \mathfrak{so}_7)$	0
33	$(\mathfrak{so}_7, \mathfrak{so}_7)$	0
34	$(2\mathfrak{sl}_2, 2\mathfrak{sl}_2)$	0
35	$(\mathfrak{sl}_3, \mathfrak{sl}_3)$	0
36	(G_2, G_2)	0

Notons que l'orbite 15 de EVIII n'apparaît pas dans [PT], elle est néanmoins bien $\mathfrak{p}$-distinguée. Notons aussi que l'orbite 1 de EIV n'est pas $\mathfrak{p}$-distinguée suite à une erreur dans la table VII de [Dj2] mentionnée par King dans [Ki].

Conséquence. Les algèbres de Lie simples symétriques GI, FI, FII, EIII, EVI, EVII et EIX ne possèdent pas d'élément presque $\mathfrak{p}$-distingué non $\mathfrak{p}$-distingué donc en vertu de la proposition 1.4.1, la conjecture 1.4.2 est démontrée dans ces cas-là.

6.2 Quelques réductions

Dans une algèbre de Lie simple symétrique donnée, on note $\mathcal{O}_i$ l'orbite de numéro i des tables de D.Z. Djokovic. Énumérons quelques réductions d'éléments presque $\mathfrak{p}$-distingués.

Reference	Type de $\mathfrak{g}$	n°	dim $\mathcal{O}$	$\delta(e)$
[Dj5]	EII	22	29	1
		24	30	0
[Dj6]	EV	50	52	2
		54	53	1
		81	59	1
		85	60	0
[Dj7] Table 2 et Fig.2	EVIII	81	107	1
		84	108	0
		88	109	1
		91	110	0
		95	111	1
		98	112	0
[Dj5] Fig.2	EI	21	34	1
		18	35	0
		17	32	1
		22	33	0

Il ne reste que 5 orbites sur lesquelles nous ne pouvons pas nous prononcer pour l'instant ; elles ne possèdent pas de réduction. Ce sont les suivantes.

Type	nř de $\mathcal{O}$	K-diagramme de Dynkin d'une caractéristique de $\mathcal{O}$	G-diagramme de Dynkin d'une caractéristique de $\mathcal{O}$
EIV	1	0001	100001
EI	16	1111	111011
EVIII	85	11111111	10010101
EI	12	2002	000200
EI	23	0020	000200

6.3 Utilisation du lemme 1.4.2

Nous allons montrer que les trois premières orbites du tableau précédent n'engendrent pas de composante étrange, grâce au lemme 1.4.2. Comme dans la section 5, on étudie le h-espace propre associé à la valeur propre 1.

Soit $\mathfrak{g} = E_{6(-26)}$ (cas EIV). On s'intéresse à l'orbite $\mathcal{O}_1$ (cf. [Dj2]). On considère un $\mathfrak{sl}_2$-triplet normal standard (e, h, f) avec $e \in \mathcal{O}_1$. On peut calculer facilement les dimensions de $\mathfrak{k}(i, h)$ et $\mathfrak{p}(i, h)$ pour les différents entiers i et on obtient (cf. [JN]) :

i	$\dim \mathfrak{k}(i, h)$	$\dim \mathfrak{p}(i, h)$
> 2	0	0
2	7	1
1	8	8
0	22	8

On en déduit que tout élément non nul e de $\mathfrak{p}(2, h)$ est régulier sous l'action de $K(0, h)$ dans $\mathfrak{p}(2, h)$ et appartient donc à $\mathcal{O}_1$ (cf. [Kaw, Lemme 2.2.9]). De la même façon, comme $h' = 2h$ est une caractéristique pour l'orbite régulière $\mathcal{O}_2$, tout élément régulier sous l'action de $K(0, h') = K(0, h)$ dans $\mathfrak{p}(2, h') = \mathfrak{p}(1, h)$ appartient à $\mathcal{O}_2$. Mais comme $\mathfrak{k}(3, h) = \{0\}$, on a $\mathfrak{p}(1, h) = \mathfrak{p}(e, 1)$. Soit donc e' régulier dans $\mathfrak{p}(e, 1)$; on a $e' \in \mathcal{O}_2$ et $[e', e] = 0$, d'où $\mathcal{N} \cap \mathfrak{p}^e \not\subset \overline{\mathcal{O}_1}$, et par le lemme 1.4.2, e ne peut pas engendrer de composante étrange.

Proposition 6.3.1. *Si $(\mathfrak{g}, \mathfrak{k})$ est de type EI, il existe $e \in \mathcal{O}_{16}$ et $e_1 \in \mathcal{O}_{18}$ tels que $[e, e_1] = 0$. Si $(\mathfrak{g}, \mathfrak{k})$ est de type EVIII, il existe $e \in \mathcal{O}_{85}$ et $e_1 \in \mathcal{O}_{109}$ tels que $[e, e_1] = 0$.*

Démonstration. Soit $\mathfrak{g}$ de type E_6 (resp. E_8). Nous allons fixer une sous algèbre de Cartan $\mathfrak{h}$ de $\mathfrak{g}$, une base de Chevalley $(X_{\alpha_i})_{\alpha_i \in R(\mathfrak{g}, \mathfrak{h})}$ compatible avec le système de racine $R(\mathfrak{g}, \mathfrak{h})$, et une involution explicite de type EI (resp. EVIII) telle que $\mathfrak{h} \cap \mathfrak{k}$ soit une sous-algèbre de Cartan de $\mathfrak{k}$. Pour simplifier les notations, on notera parfois X_i à la place de X_{α_i}. Soit une caractéristique $h \in \mathfrak{h} \cap \mathfrak{k}$ de l'orbite $\mathcal{O}_{16}$ (resp. $\mathcal{O}_{85}$). Il se trouve que $h' = 2h$ est alors une caractéristique de l'orbite $\mathcal{O}_{18}$ (resp. $\mathcal{O}_{109}$). Avec les notations de la section 1.2, on rappelle que l'on a $\mathfrak{k}(0, h) = \mathfrak{k}(0, h')$ et $\mathfrak{p}(2, h') = \mathfrak{p}(1, h)$. De plus, d'après [Kaw, Lemme 2.2.9], $\mathcal{O}_{16} \cap \mathfrak{p}(2, h)$ (resp. $\mathcal{O}_{85} \cap \mathfrak{p}(2, h)$) est exactement la $K(0, h)$-orbite régulière dense de $\mathfrak{p}(2, h)$, tandis que $\mathcal{O}_{18} \cap \mathfrak{p}(1, h)$ (resp. $\mathcal{O}_{109} \cap \mathfrak{p}(1, h)$) est la $K(0, h)$-orbite régulière dense de $\mathfrak{p}(1, h)$. Grâce aux tables de [JN], on voit que pour tout $e \in \mathcal{O}_{16} \cap \mathfrak{p}(2, h)$, on a $\dim \mathfrak{p}(e, 1) = 1$. Nous choisirons donc un élément régulier $e \in \mathfrak{p}(e, 2)$ et montrer qu'il commute avec un élément e_1, régulier dans $\mathfrak{p}(1, h)$.

Commençons par le premier cas. On fixe une base $(\alpha_i)_{1 \leqslant i \leqslant 6}$ telle que le diagramme de Dynkin de $\mathfrak{g}$ s'écrive

$$\begin{array}{ccccc} \alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ & & \alpha_2 & & \end{array} .$$

On choisit la numérotation des racines donnée par [Dj3, Table 3] ainsi que la base de Chevalley de [Dj3, §4] caractérisée par [Dj3, Table 13]. Ainsi, on a par exemple $\alpha_8 = \alpha_2 + \alpha_4$ et $[X_2, X_4] = -X_8$. On fixe l'involution τ, définie dans [Dj2] (dans [Dj2] les racines $\alpha_2, \alpha_3, \alpha_4$ sont notées respectivement $\alpha_4, \alpha_2, \alpha_3$), permutant les racines α_1 et α_6, α_3 et α_5 et laissant invariants α_2 et α_4. Pour trouver une involution θ de type EI, on peut poser (cf. [Dj2])

$$\theta(X_{\sum_{i=1}^6 k_i\alpha_i}) = (-1)^{k_2} X_{\sum_{i=1}^6 k_i\tau(\alpha_i)}.$$

Toujours selon [Dj2] on définit

$$\beta_1 = (\alpha_1)_{|\mathfrak{h}\cap\mathfrak{k}} = (\alpha_6)_{|\mathfrak{h}\cap\mathfrak{k}}, \quad \beta_2 = (\alpha_3)_{|\mathfrak{h}\cap\mathfrak{k}} = (\alpha_5)_{|\mathfrak{h}\cap\mathfrak{k}}, \quad \beta_3 = (\alpha_4)_{|\mathfrak{h}\cap\mathfrak{k}},$$
$$\beta_4 = (\alpha_2)_{|\mathfrak{h}\cap\mathfrak{k}}, \qquad \beta_0 = -2\beta_1 - 3\beta_2 - 2\beta_3 - \beta_4.$$

De [Dj2, Table VIII], on déduit que l'élément h vérifiant $\beta_1(h) = \beta_2(h) = \beta_3(h) = \beta_0(h) = 1$ est une caractéristique de $\mathcal{O}_{16}$. Si on écrit le G-diagramme de Dynkin de h, on trouve

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ & & -8 & & \end{matrix}.$$

Ceci nous permet d'obtenir les bases suivantes pour $\mathfrak{k}(0,h)$, $\mathfrak{p}(1,h)$, $\mathfrak{p}(2,h)$ et $\mathfrak{k}(3,h)$:

$\mathfrak{k}(0,h)$	$H_1 + H_6; H_3 + H_5; H_2; H_4$
$\mathfrak{p}(1,h)$	$A_1 := X_1 - X_6; A_2 := X_3 - X_5; A_3 := X_{-32} + X_{-33}; A_4 := X_{35}$
$\mathfrak{p}(2,h)$	$B_1 := X_7 - X_{11}; B_2 := X_9 - X_{10}; B_3 := X_{-29} + X_{-31}; B_4 := X_{-30}$
$\mathfrak{k}(3,h)$	$C_1 := X_{12} + X_{16}; C_2 := X_{-26} - X_{-28}; C_3 := X_{15}$

Comme $\mathfrak{k}(0,h) \subset \mathfrak{h}$, un élément $x \in \mathfrak{h}$ agit par multiplication par un scalaire sur chaque A_i et chaque B_j. En particulier, un élément $e = \sum b_i B_i$ (resp. $e_1 = \sum a_i A_i$) est régulier dans $\mathfrak{p}(2,h)$ (resp. $\mathfrak{p}(1,h)$) si et seulement si $[\mathfrak{k}(0,h), e] = \mathfrak{p}(2,h)$ (resp. $[\mathfrak{k}(0,h), e_1] = \mathfrak{p}(1,h)$) ce qui équivaut au fait que les coefficients b_i (resp. a_i) sont tous non nuls. On choisit $e = B_1 + B_2 + B_3 + B_4$. Ecrivons la table de multiplication dans la matrice ci-dessous, où l'élément d'indice (i,j) correspond à $[A_i, B_j]$:

$$([A_i, B_j])_{i,j} = \begin{pmatrix} 0 & C_1 & 0 & C_2 \\ 0 & -2C_3 & -C_2 & 0 \\ C_2 & 0 & 0 & 0 \\ 0 & 0 & C_1 & C_3 \end{pmatrix}$$

D'après ce qui précède, l'élément $e_1 = -2A_1 + A_2 + 3A_3 + 2A_4$ est régulier dans $\mathfrak{p}(1,h)$, appartient à $\mathcal{O}_{18}$ et commute avec e.

Le second cas où $(\mathfrak{g}, \mathfrak{k})$ est de type EVIII se traite de façon similaire. On fixe la base de $R(\mathfrak{g}, \mathfrak{h})$:

$$\begin{matrix} \alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\ & & \alpha_2 & & & & \end{matrix}.$$

La numérotation des racines et la base de Chevalley sont données par [Dj4, Table 1, Table 6]. Conformément à [Dj1], on pose

$$\theta(X_{\sum_{i=1}^{8} k_i\alpha_i}) = (-1)^{k_1} X_{\sum_{i=1}^{8} k_i\alpha_i}.$$

On peut choisir la caractéristique h de $\mathcal{O}_{85}$ de sorte que son G-diagramme de Dynkin soit

$$\begin{matrix} -14 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & 1 & & & & \end{matrix}.$$

On obtient facilement des bases pour les espaces de poids suivant :

$\mathfrak{k}(0,h)$	$H_1; H_2; H_3; H_4; H_5; H_6; H_7; H_8$
$\mathfrak{p}(1,h)$	$A_1 := X_{93}; A_2 := X_{94}; A_3 := X_{95}; A_4 := X_{96}$ $A_5 := X_{-85}; A_6 := X_{-86}; A_7 := X_{-87}; A_8 := X_{-88}$
$\mathfrak{p}(2,h)$	$B_1 := X_{98}; B_2 := X_{99}; B_3 := X_{100}; B_4 := X_{-80}$ $B_5 := X_{-81}; B_6 := X_{-82}; B_7 := X_{-83}; B_8 := X_{-84}$
$\mathfrak{k}(3,h)$	$C_1 := X_{17}; C_2 := X_{18}; C_3 := X_{19}; C_4 := X_{20}$ $C_5 := X_{21}; C_6 := X_{22}; C_7 := X_{-118}$

Ici encore $\mathfrak{k}(0,h) \subseteq \mathfrak{h}$, donc les orbites régulières de $\mathfrak{p}(1,h)$ et $\mathfrak{p}(2,h)$ se décrivent de la même façon que dans le cas EI. La table de multiplication : $\mathfrak{p}(1,h) \times \mathfrak{p}(2,h) \to \mathfrak{k}(3,h)$ est la suivante :

$$([A_i, B_j])_{i,j} = \begin{pmatrix} 0 & 0 & 0 & -C_3 & 0 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_4 & 0 & C_2 & 0 \\ 0 & 0 & 0 & 0 & C_5 & 0 & 0 & C_2 \\ 0 & 0 & 0 & 0 & 0 & C_6 & C_5 & C_4 \\ 0 & 0 & -C_6 & 0 & 0 & 0 & -C_7 & 0 \\ C_3 & 0 & -C_5 & 0 & 0 & -C_7 & 0 & 0 \\ C_1 & 0 & 0 & -C_7 & 0 & 0 & 0 & 0 \\ 0 & -C_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Finalement, on peut poser

$$e = \sum_{i=1}^{8} B_i \text{ et } e_1 = A_1 + 2A_2 + 3A_3 - 2A_4 - 2A_5 + A_6 + A_7 + 5A_8.$$

Ces deux éléments commutent et appartiennent respectivement à $\mathcal{O}_{85}$ et $\mathcal{O}_{109}$. □

Corollaire 6.3.1. *Si $(\mathfrak{g}, \mathfrak{k})$ est de type EI, $\mathfrak{C}(\mathcal{O}_{18})$ n'engendre pas de composante étrange. Même conclusion pour $\mathcal{O}_{85}$ si $(\mathfrak{g}, \mathfrak{k})$ est de type EVIII.*

Démonstration. Il suffit d'appliquer la proposition précédente et le lemme 1.4.2. □

Terminons cette section en donnant le nombre de composantes irréductibles de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$ dans les différents cas exceptionnels.

GI	FI	FII	EI	EII	EIII	EIV	EV	EVI	EVII	EVIII	EIX
3	10	2	4 à 6	17	8	1	27	17	11	33	16

Conclusion

La conjecture A est démontrée dans les cas suivants : AIII, CII, DIII, EII, EIII, EIV, EV, EVI, EVII, EVIII, EIX, FI, FII, GI et dans les cas

AI $(\mathfrak{sl}_n, \mathfrak{so}_n)$ avec $n \leqslant 5$ ($i.e.$ en rang $\leqslant 4$)

AII $(\mathfrak{sl}_{2n}, \mathfrak{sp}_{2n})$ avec $n \leqslant 3$ ($i.e$ en rang $\leqslant 3$).

BDI $(\mathfrak{so}_n, \mathfrak{so}_p \times \mathfrak{so}_q)$ avec $p \leqslant 2$ ou $q \leqslant 2$ ou $max(p,q) \leqslant 4$. (En particulier, elle est vraie en rang $\leqslant 2$)

CI $(\mathfrak{sp}_{2n}, \mathfrak{gl}_n)$ avec $n \leqslant 7$. ($i.e.$ en rang $\leqslant 7$)

De plus, les méthodes de réduction et d'étude de $\mathfrak{p}(e,1)$ introduites dans les sections 4 et 5 montrent qu'un certain nombre d'éléments presque $\mathfrak{p}$-distingués ne fournissent pas de composante étrange.

Nous avons exploité au maximum les deux outils dont nous nous sommes dotés : d'une part le lemme 1.4.2 et d'autre part la réduction dont le principe est contenu dans le lemme 4.0.1. En effet, les seules orbites pour lesquelles nous ne savons rien dire sont des orbites $\mathcal{O}$ contenant des éléments e presque $\mathfrak{p}$-distingués, non $\mathfrak{p}$-distingués, vérifiant $\mathfrak{p}^e \cap \mathcal{N} \subseteq \overline{\mathcal{O}}$, et n'ayant pas de réduction.

7 Appendice : Orbites quasi-$\mathfrak{p}$-distinguées

Le lemme 1.4.2 nous a invité à déterminer les éléments nilpotents e vérifiant $\mathcal{N}(\mathfrak{p}^e) \subset \overline{K.e}$. Dans le cas des algèbres de Lie, D. Panyushev a dernièrement nommé ces éléments *self-large* dans [Pa5]. Ils sont caractérisés de la façon suivante : ce sont les éléments presque distingués e vérifiant $\mathfrak{g}(e,1) = 0$. N'ayant découvert ces travaux que récemment, nous allons présenter dans cet appendice quelques résultats inspirés de [Pa5] qui permettent de déterminer des éléments similaires dans les algèbres de Lie symétriques.

Définition 7.0.1. Un élément nilpotent $e \in \mathfrak{p}$ vérifiant $\mathcal{N}(\mathfrak{p}^e) \subset \overline{K.e}$ est dit *quasi-$\mathfrak{p}$-distingué*. L'orbite $K.e$ est alors également appelée quasi-$\mathfrak{p}$-distinguée.

Il résulte du lemme 1.4.2 que seuls des éléments quasi-$\mathfrak{p}$-distingués pouvaient engendrer une composante irréductible de $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{p})$. On a en particulier établi les implications suivantes :

$$\mathfrak{p}\text{-distingué} \Rightarrow \text{quasi-}\mathfrak{p}\text{-distingué} \Rightarrow \text{presque } \mathfrak{p}\text{-distingué}.$$

Comme précédemment, on fixe un $\mathfrak{sl}_2$-triplet (e,h,f), ce qui nous donne les graduations suivantes de $\mathfrak{w} = \mathfrak{g}, \mathfrak{k}$ ou $\mathfrak{p}$:

$$\mathfrak{w} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{w}(i,h); \qquad \mathfrak{w}^e = \bigoplus_{i \geqslant 0} \mathfrak{w}(e,i); \qquad \mathfrak{w}^f = \bigoplus_{i \leqslant 0} \mathfrak{w}(f,i).$$

On sait que $\mathfrak{p}(e,0)$ agit (via l'action adjointe de $\mathfrak{g}$) sur $\mathfrak{g}(f,-1)$. Si e est presque $\mathfrak{p}$-distingué, $\mathfrak{p}(e,0)$ est un tore, on peut alors considérer $\mathfrak{X}(\mathfrak{p}(e,0))$ l'ensemble des poids non-nuls du $\mathfrak{p}(e,0)$-module $\mathfrak{g}(f,-1)$. On notera la décomposition en espaces de poids comme ceci :

$$\mathfrak{g}(f,-1) = \bigoplus_{\gamma \in \mathfrak{X}(\mathfrak{p}(e,0))} V_\gamma.$$

Le but de la proposition 7.0.2 est de donner un analogue faible de [Pa5, Théorème 2.1] pour les algèbres de Lie symétriques. Il se trouve que, combinée aux résultats des sections précédentes, cette proposition va s'avérer suffisante pour décrire toutes les orbites quasi-$\mathfrak{p}$-distinguées des algèbres de Lie symétriques. Nous avons tout d'abord besoin de deux lemmes. On rappelle que L désigne la forme de Killing sur $\mathfrak{g}$.

Lemme 7.0.2 (D. Panyushev [Pa5])**.** *L'application*

$$\Phi : \left\{ \begin{array}{rcl} \mathfrak{g}(f,-1)\times\mathfrak{g}(f,-1) & \to & \Bbbk \\ (\xi,\eta) & \mapsto & L(e,[\xi,\eta]) \end{array} \right.$$

est une forme bilinéraire antisymétrique non dégénérée $\mathfrak{g}(e,0)$-invariante.

Lemme 7.0.3. *L'automorphisme θ induit une bijection entre V_γ et $V_{-\gamma}$. L'application $\tilde{\Phi}$: $(\xi,\eta)\mapsto\Phi(\xi,\theta(\eta))$ est une forme bilinéaire symétrique non dégénérée. Elle reste non dégénérée sur chaque sous-espace V_γ.*

Démonstration. Comme $h\in\mathfrak{k}$ et $f\in\mathfrak{p}$, l'automorphisme θ induit une bijection de $\mathfrak{g}(f,-1)$. Soit ξ un élément de V_γ. Pour tout $t\in\mathfrak{p}(e,0)$, on a

$$[t,\theta(\xi)] = \theta([\theta(t),\xi]) = \theta(-\gamma(t)\xi) = -\gamma(t)\theta(\xi).$$

Ceci prouve la première affirmation.

Comme θ induit une bijection de $\mathfrak{g}(f,-1)$, le fait que $\tilde{\Phi}$ soit non-dégénérée est une conséquence du lemme 7.0.2. Vérifions que $\tilde{\Phi}$ est symétrique :

$$\begin{array}{rcl} L(e,[\xi,\theta(\eta)]) & = & L(\theta(e),\theta([\xi,\theta(\eta)])) \\ & = & L(-e,[\theta(\xi),\eta]) \\ & = & L(e,[\eta,\theta(\xi)]). \end{array}$$

Prouvons enfin la dernière affirmation du lemme. Soit $\xi\in V_\gamma$ et $\eta\in V_\mu$, alors $\theta(\eta)\in V_{-\mu}$ et pour tout $t\in\mathfrak{p}(e,0)$, on a

$$(\gamma(t)-\mu(t))\tilde{\Phi}(\xi,\eta) = \Phi([t,\xi],\theta(\eta)) + \Phi(\xi,[t,\theta(\eta)]) = 0.$$

donc $\tilde{\Phi}(\xi,\eta)=0$ si $\gamma\neq\mu$. Ceci montre que $\tilde{\Phi}$ est non-dégénérée sur V_γ. $\square$

Proposition 7.0.2. *Si e est presque $\mathfrak{p}$-distingué tel que $\mathfrak{g}(f,-1)^{\mathfrak{p}(e,0)} = \{0\}$ et $\mathfrak{p}(e,1) \neq \{0\}$, alors e n'est pas quasi-$\mathfrak{p}$-distingué.*

Démonstration. Notons tout d'abord que, d'après les hypothèses, $0 \notin \mathfrak{X}(\mathfrak{p}(e,0))$. Fixons $\mu \in \mathfrak{X}(\mathfrak{p}(e,0))$ et soit $\xi \in V_\mu$ tel que $\tilde{\Phi}(\xi,\xi) \neq 0$. Puisque $\mu \neq 0$, il existe $t \in \mathfrak{p}(e,0)$ tel que $[t,\xi] = \xi$ et $[t,\theta(\xi)] = -\theta(\xi)$. Un calcul facile donne

$$L([[e,(\xi+\theta(\xi))],(\xi+\theta(\xi))],t) = 2L(e,[\xi,\theta(\xi)]) \neq 0,$$

ce qui montre en particulier que $[[e,\xi+\theta(\xi)],\xi+\theta(\xi)] \neq 0$. Donc $z = \xi+\theta(\xi)$ est un élément de $\mathfrak{k}(f,-1)$ satisfaisant $[[e,z],z] \neq 0$. Finalement, par [Pa5, Lemme 2.3], $G.(e+[z,e])$ est une orbite strictement plus grande que $G.e$, ce qui implique notamment que $K.(e+[z,e]) \not\subseteq \overline{K.e}$. Pour conclure, il est facile de vérifier que $e+[z,e] \in \mathfrak{p}^e$. □

Remarque 7.0.4. On ne peut pas supprimer l'hypothèse $\mathfrak{g}(f,-1)^{\mathfrak{p}(e,0)} = \{0\}$. En effet, l'orbite $\mathcal{O}_1$ de FII vérifie $\mathfrak{p}(e,0) = \{0\}$ et $\mathfrak{p}(e,1) \neq \{0\}$. Elle est cependant $\mathfrak{p}$-distinguée, donc quasi-$\mathfrak{p}$-distinguée.

Corollaire 7.0.5. *Les orbites $\mathcal{O}_{50}$ de EV; $\mathcal{O}_{85}$, $\mathcal{O}_{88}$ de EVIII et $\mathcal{O}_{16}$, $\mathcal{O}_{17}$ de EI ne sont pas quasi-$\mathfrak{p}$-distinguées.*

Démonstration. On a vu dans la section 6 que ces orbites nilpotentes sont presque $\mathfrak{p}$-distinguées et qu'elles ne sont pas paires. Grâce à [Dj1, Dj2], on peut calculer $\mathfrak{p}(e,0)$ et on trouve que $\mathfrak{p}(e,0) = \mathfrak{g}(e,0)$ dans ces cas précis. On peut maintenant appliquer l'argument suivant de [Pa5]. Soit $\mathfrak{l} = \mathfrak{g}^{\mathfrak{g}(e,0)}$ et $\mathfrak{s} = [\mathfrak{l},\mathfrak{l}]$, de sorte que $\mathfrak{l} = \mathfrak{s} \oplus \mathfrak{g}(e,0)$. Alors e est distingué dans $\mathfrak{s}$ et la graduation induite par $h \in \mathfrak{s}$ vérifie $\{0\} = \mathfrak{s}(-1,h) = \mathfrak{l}(-1,h) = \mathfrak{g}(-1,h)^{\mathfrak{g}(e,0)} = \mathfrak{g}(-1,h)^{\mathfrak{p}(e,0)} \supseteq \mathfrak{g}(f,-1)^{\mathfrak{p}(e,0)}$. En combinant ceci avec les tables de [JN], on montre que les hypothèses de la proposition 7.0.2 sont satisfaites. Les orbites mentionnées ne sont donc pas quasi-$\mathfrak{p}$-distinguées. □

En suivant la même idée, il est possible de donner une preuve alternative de la description de la section 5.2 des orbites presque $\mathfrak{p}$-distinguées de AI qui ne sont pas quasi-$\mathfrak{p}$-distinguées. Malheureusement, la proposition 7.0.2 ne permet pas l'étude des orbites presque $\mathfrak{p}$-distinguées non quasi-$\mathfrak{p}$-distinguée de AII ni de l'orbite $\mathcal{O}_1$ de EIV. Pour ces orbites, on se réfère aux sections 5.3 et 6.3. Finalement, les dernières orbites à traiter étant paires, on peut lister l'ensemble des orbites quasi-$\mathfrak{p}$-distinguées dans les différents cas simples à l'aide du lemme 1.4.3. Le cas général s'en déduit facilement étant donné que les algèbres de Lie symétriques sont produit direct d'algèbres de Lie symétriques simples.

Cas	Orbites quasi-$\mathfrak{p}$-distinguées.
AI	L'orbite ($\mathfrak{p}$-distinguée) régulière et les orbites dont le diagramme associé est constituée de lignes de longueurs différant d'au moins 2.
AII	L'orbite ($\mathfrak{p}$-distinguée) régulière et les orbites dont le diagramme associé est constituée de paires de lignes de longueurs différant d'au moins 2.
AIII	Les orbites $\mathfrak{p}$-distinguées (*i.e.* qui ont un ab-diagramme dont les lignes de même longueur débutent par la même lettre), ce sont les seules orbites presque $\mathfrak{p}$-distinguées.
BDI, CI	Les orbites presque $\mathfrak{p}$-distinguées (cf. section 3.3).
CII, DIII	Les orbites $\mathfrak{p}$-distinguées (qui sont les seules orbites presque $\mathfrak{p}$-distinguées, cf. section 3.3).
EIII, EVI, EVII, EIX, FI, FII, GI	Les orbites $\mathfrak{p}$-distinguées (qui sont les seules orbites presque $\mathfrak{p}$-distinguées, cf. section 6.1).
EII	Les orbites presque $\mathfrak{p}$-distinguées. En particulier l'orbite $\mathcal{O}_{22}$ non $\mathfrak{p}$-distinguée.
EIV	L'orbite régulière ($\mathfrak{p}$-distinguée).
EI	Les orbites $\mathfrak{p}$-distinguées et les orbites $\mathcal{O}_{12}, \mathcal{O}_{21}, \mathcal{O}_{23}$ (cf. section 6 et lemme 1.4.3).
EV	Les orbites $\mathfrak{p}$-distinguées et l'orbite $\mathcal{O}_{81}$ (cf. section 6 et lemme 1.4.3).
EVIII	Les orbites $\mathfrak{p}$-distinguées et les orbites $\mathcal{O}_{81}, \mathcal{O}_{95}$ (cf. section 6 et lemme 1.4.3).
Algèbres de Lie	Les orbites dont les éléments e vérifient $\mathfrak{p}(e,0)$ est un tore et $\mathfrak{p}(e,1) = \{0\}$ (cf. [Pa5, Théoreme 2.1]).

Les calculs permettent de montrer le fait suivant. Les éléments quasi-$\mathfrak{p}$-distingués d'une algèbres de Lie symétrique simple sont exactement les éléments e vérifiant l'une des conditions suivantes

- $\mathfrak{p}(e,0) = \{0\}$ *i.e.* e est $\mathfrak{p}$-distingué ;
- $\mathfrak{p}(e,0)$ est un tore et $\mathfrak{p}(e,1) = \{0\}$.

Cependant, ceci est uniquement valable dans le cas simple. En effet, la remarque 7.0.4 implique que, dans FII×EI, l'orbite $\mathcal{O}_1 \times \mathcal{O}_{21}$ est quasi-$\mathfrak{p}$-distinguée mais vérifie $\mathfrak{p}(e,0) = T_1 \neq \{0\}$ et $\mathfrak{p}(e,1) \neq \{0\}$.

Chapitre II

Sheets

Introduction

Let $\mathfrak{g}$ be a finite dimensional reductive Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero. Fix an involutive automorphism θ of $\mathfrak{g}$; it yields an eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ associated to respective eigenvalues $+1$ and -1. One then says that $(\mathfrak{g}, \theta)$, or $(\mathfrak{g}, \mathfrak{k})$, is a symmetric Lie algebra, or a symmetric pair. Denote by G the adjoint group of $\mathfrak{g}$ and by $K \subset G$ the connected subgroup with Lie algebra $\mathfrak{k} \cap [\mathfrak{g}, \mathfrak{g}]$. The adjoint action of $g \in G$ on $x \in \mathfrak{g}$ is denoted by $g.x$. Recall that a *G-sheet* of $\mathfrak{g}$ is an irreducible component of $\mathfrak{g}^{(m)} := \{x \in \mathfrak{g} \mid \dim G.x = m\}$ for some $m \in \mathbb{N}$. This notion can be obviously generalized to $(\mathfrak{g}, \theta)$: the *K-sheets* of $\mathfrak{p}$ are the irreducible components of the $\mathfrak{p}^{(m)} := \{x \in \mathfrak{p} \mid \dim K.x = m\}$, $m \in \mathbb{N}$. The study of these varieties is related to various geometric problems occuring in Lie theory. For example, the study of the irreducibility of the commuting variety in [Ri1] and of its symmetric analogue in [Pa4, SY2, PY] is based on some results about G-sheets and K-sheets.

Let us first recall some results about G-sheets. The G-sheets containing a semisimple element are called Dixmier sheets; they were introduced by Dixmier in [Di1, Di2]. Any G-sheet is Dixmier when $\mathfrak{g} = \mathfrak{gl}_N$; in [Kr], Kraft gave a parametrization of conjugacy classes of sheets in this case. Bohro and Kraft introduced in [BK] the notion of sheet for an arbitrary representation, which includes the above definitions of G-sheets and K-sheets. They also generalized in [Boh, BK] some of the results of [Kr] to any semisimple $\mathfrak{g}$. This parametrization relies on the *induction of nilpotent orbits*, defined by Lusztig-Spaltenstein [LS], and the notion of *decomposition classes* or *Zerlegungsklassen*. Following [TY, 39.1], a decomposition class will here be called a *Jordan G-class*. The Jordan G-class of an element $x \in \mathfrak{g}$ can be defined by

$$J_G(x) := \{y \in \mathfrak{g} \mid \exists\, g \in G,\ g.\mathfrak{g}^x = \mathfrak{g}^y\}$$

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(where $\mathfrak{g}^x$ is the centralizer of x in $\mathfrak{g}$). Clearly, Jordan G-classes are equivalence classes and one can show that $\mathfrak{g}$ is a finite disjoint union of these classes. Then, it is easily seen that a G-sheet is the union of Jordan G-classes. A significant part of the work made in [Boh, BK] consists in characterizing a G-sheet by the Jordan G-classes it contains. Basic results on Jordan classes (finiteness, smoothness, description of closures,...) can be found in [TY, Chapter 39] and one can refer to Broer [Bro] for more advanced properties (geometric quotients, normalisation of closure,...).
An important example of a G-sheet is the *set of regular elements*:

$$\mathfrak{g}^{reg} = \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x \leqslant \dim \mathfrak{g}^y \text{ for all } y \in \mathfrak{g}\}.$$

Kostant [Ko] has shown that the geometric quotient $\mathfrak{g}^{reg}/G$ exists and is isomorphic to an affine space. This has been generalized to the so-called *admissible* G-sheets in [Ru]. Then, Katsylo proved in [Kat] the existence of a *geometric quotient* S/G for any G-sheet S. More recently, Im Hof [IH] showed that the G-sheets are *smooth* when $\mathfrak{g}$ is of classical type.
The parametrization of sheets used in [Ko, Ru, Kat, IH] differs from the one given in [Kr, Boh, BK] by the use of "Slodowy slices". More precisely, let S be a sheet containing the nilpotent element e and embed e into an $\mathfrak{sl}_2$-triple (e, h, f). Following the work of Slodowy [Sl, §7.4], the associated *Slodowy slice* $e + X$ of S is defined by

$$e + X := (e + \mathfrak{g}^f) \cap S.$$

Then, one has $S = G.(e + X)$ and S/G is isomorphic to the quotient of $e + X$ by a finite group [Kat]. Furthermore, since the morphism $G \times (e + X) \to S$ is smooth [IH], the geometry of S is closely related to that of $e + X$. We give a more detailed presentation of these results in the first section.

In the symmetric case, much less properties of sheets are known. The first important one was obtained in [KR] where the *regular sheet* $\mathfrak{p}^{reg}$ of $\mathfrak{p}$ is studied. In particular, similarly to [Ko], it is shown that $\mathfrak{p}^{reg} = G^\theta.(e^{reg} + \mathfrak{p}^f)$ where $G^\theta = \{g \in G \mid g \circ \theta = \theta \circ g\}$. Another interesting result is obtained in [Pa4, SY2, PY] (where the symmetric commuting variety is studied): each even nilpotent element of $\mathfrak{p}$ belongs to some K-sheet containing a semisimple element. More advanced results can be found in [TY, §39]. The *Jordan K-class* of $x \in \mathfrak{p}$ is defined by

$$J_K(x) := \{y \in \mathfrak{p} \mid \exists\, k \in K,\ k.\mathfrak{p}^x = \mathfrak{p}^y\}.$$

One can find in [TY] some properties of Jordan K-classes (finiteness, dimension, ...) and it is shown that a K-sheet is a finite disjoint union of such classes.
Unfortunately, the key notion of "orbit induction" does not seem to be well adapted to the symmetric case. For instance, the definition introduced by Ohta in [Oh3] does not leave invariant the orbit dimension anymore.

We now turn to the results of this paper. The inclusion $\mathfrak{p}^{(m)} \subset \mathfrak{g}^{(2m)}$ is the starting point for studying the intersection of G-sheets, or Jordan classes, with $\mathfrak{p}$ in order to get some information about K-sheets.

We first consider the case of symmetric pairs of type 0 in section 2.1. A symmetric pair is said to be of type 0 if it is isomorphic to a pair $(\mathfrak{g}' \times \mathfrak{g}', \theta)$ with $\theta(x, y) = (y, x)$. This case, often called the "group case", is the symmetric analogue of the Lie algebra $\mathfrak{g}'$. As expected, we show that the K-sheets of $\mathfrak{p}$ are in one to one correspondence with the G'-sheets of $\mathfrak{g}'$.
In the general case we study the intersection $J \cap \mathfrak{p}$ when J is a Jordan G-class. Using the results obtained in sections 2.2 to 2.4, we show (see Theorem 2.4.4) that $J \cap \mathfrak{p}$ is smooth, equidimensional, and that its irreducible components are exactly the Jordan K-classes it contains.
We study the K-sheets, for a general symmetric pair, in section 2.5. After proving the smoothness of K-sheets in classical cases (Remark 2.5.4), we try to obtain a parametrization similar to the Lie algebra case by using generalized "Slodowy slices" of the form $e + X \cap \mathfrak{p}$, where $e \in \mathfrak{p}$ is a nilpotent element contained in the G-sheet S. To get this parametrization we need to introduce three conditions (labelled by ($\heartsuit$), ($\diamondsuit$) and ($\clubsuit$)) on the sheet S. Under these assumptions, we obtain the parametrization result in Theorem 2.5.11; it gives in particular the equidimensionality of $S \cap \mathfrak{p}$.

In the third section we show that the conditions ($\heartsuit$), ($\diamondsuit$), ($\clubsuit$) hold when $\mathfrak{g} = \mathfrak{gl}_N$ or $\mathfrak{sl}_N$ (type A). In this case, up to conjugacy, three types of irreducible symmetric pairs exist (AI, AII, AIII in the notation of [He1]) and have to be analyzed in details. The most difficult one being type AIII, i.e. $(\mathfrak{g}, \mathfrak{k}) \cong (\mathfrak{gl}_N, \mathfrak{gl}_p \times \mathfrak{gl}_{N-p})$.

In Section 4 we prove the main result in type A (Theorem 4.1.2), which gives a complete description of the K-sheets and of the intersections of G-sheets with $\mathfrak{p}$. In particular, we give the dimension of a K-sheet in terms of the dimension of the nilpotent K-orbits contained in the sheet. One can also determine the sheets which contain semisimple elements (i.e. the *Dixmier K-sheets*) and characterize nilpotent orbits which are K-sheets (i.e. the *rigid nilpotent K-orbits*).

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1 Generalities

1.1 Notation

We fix an algebraically closed field $\Bbbk$ of characteristic zero and we set $\Bbbk^\times = \Bbbk \smallsetminus \{0\}$. If V, V' are $\Bbbk$-vector spaces, $\mathrm{Hom}(V, V')$ is the vector space of $\Bbbk$-linear maps from V to V' and the dual of V is $V^* = \mathrm{Hom}(V, \Bbbk)$. The space $\mathfrak{gl}(V) = \mathrm{Hom}(V, V)$ inherits a natural Lie algebra structure by setting $[x, y] = x \circ y - y \circ x$ for $x, y \in \mathfrak{gl}(V)$. The action of $x \in \mathfrak{gl}(V)$ on $v \in V$ is written $x.v = x(v)$ and ${}^t x$ is the transpose linear map of x. If M is a subset of $\mathrm{Hom}(V, V')$ we set $\ker M = \bigcap_{\alpha \in M} \ker \alpha$.
If $\mathbf{v} = (v_1, \ldots, v_N)$ is a basis of V, the algebra $\mathfrak{gl}(V)$ can be identified with $\mathfrak{gl}(\mathbf{v}) = \mathfrak{gl}_N = \mathrm{M}_N(\Bbbk)$ (the algebra of $N \times N$ matrices). When $\mathbf{v}' = (v_{i_1}, \ldots, v_{i_k})$ is a sub-basis of $\mathbf{v}$, we may identify

$\mathfrak{gl}(\mathbf{v}')$ with a subalgebra of $\mathfrak{gl}(V)$ by extending $x \in \mathfrak{gl}(\mathbf{v}')$ as follows: $x.v_i = x.v_{i_j}$ if $i = i_j$ for some $j \in [\![1, k]\!]$, $x.v_i = 0$ otherwise.

All the varieties considered will be algebraic over $\Bbbk$ and we (mostly) adopt notations and conventions of [Har] or [TY] for relevant algebraic and topological notions. In particular, $\Bbbk[X]$ is the ring of globally defined algebraic functions on an algebraic variety X. Recall that when V is a finite dimensional vector space one has $\Bbbk[V] = S(V^*)$, the symmetric algebra of V^*.

We will refer to [TY] for most of the classical results concerning Lie algebras. As said in the introduction, $\mathfrak{g}$ denotes a finite dimensional *reductive* Lie $\Bbbk$-algebra. We write $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g})$ where $\mathfrak{z}(\mathfrak{g})$ is the centre of $\mathfrak{g}$ and we denote by $\mathrm{ad}_{\mathfrak{g}}(x) : y \mapsto [x, y]$ the adjoint action of $x \in \mathfrak{g}$ on $y \in \mathfrak{g}$. Let G be the connected algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie algebra $\mathrm{Lie}\, G = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \cong [\mathfrak{g}, \mathfrak{g}]$. The group G is called the *adjoint group* of $\mathfrak{g}$. The adjoint action of $g \in G$ on $y \in \mathfrak{g}$ is denoted by $g.y = \mathrm{Ad}(g).y$; thus, $G.y$ is the (adjoint) orbit of y.
We will generally denote Lie subalgebras of $\mathfrak{g}$ by small german letters (e.g. $\mathfrak{l}$) and the smallest algebraic subgroup of G whose Lie algebra contains $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{l})$ by the corresponding capital roman letter (e.g. L). When $\mathfrak{l}$ is an algebraic subalgebra of $\mathfrak{g}$ the subgroup L acts on $\mathfrak{l}$ as its adjoint algebraic group, cf. [TY, 24.8.5]. We denote by H° the identity component of an algebraic group H.

Let $E \subset \mathfrak{g}$ be an arbitray subset. If $\mathfrak{l}$, resp. L, is a subalgebra of $\mathfrak{g}$, resp. algebraic subgroup of G, we define the associated centralizers and normalizers by:

$$\mathfrak{l}^E = \mathfrak{c}_{\mathfrak{l}}(E) = \{x \in \mathfrak{l} \mid [x, E] = (0)\}, \quad L^E = Z_L(E) = C_G(E) = \{g \in L \mid g.x = x \text{ for all } x \in E\},$$
$$N_L(E) = \{g \in L \mid g.E \subset E\}.$$

When $E = \{x\}$ we simply write $\mathfrak{l}^x$, L^x, etc. Recall from [TY, 24.3.6] that $\mathrm{Lie}\, L^E = \mathfrak{l}^E$. As in [TY], the set of "regular" elements in E is denoted by:

$$E^\bullet = \{x \in E : \dim \mathfrak{g}^x = \min_{y \in E} \dim \mathfrak{g}^y\} = \{x \in E : \dim G.x = \max_{y \in E} \dim G.y\}. \tag{1.1}$$

Any $x \in \mathfrak{g}$ has a *Jordan decomposition* in $\mathfrak{g}$, that we will very often write $x = s + n$ (cf. [TY, 20.4.5, 20.5.9]). Thus: s is semisimple, i.e. $\mathrm{ad}_{\mathfrak{g}}(s) \in \mathfrak{gl}(\mathfrak{g})$ is semisimple, n is nilpotent, i.e. $\mathrm{ad}_{\mathfrak{g}}(n)$ is nilpotent, and $[s, n] = 0$. The element s, resp. n, is called the semisimple, resp. nilpotent, part (or component) of x. An $\mathfrak{sl}_2$*-triple* is a triple (e, h, f) of elements of $\mathfrak{g}$ satisfying the relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$; then, $\mathfrak{h} = ([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{h}) \oplus \mathfrak{z}(\mathfrak{g})$ and the *rank* of $\mathfrak{g}$ is $\mathrm{rk}\, \mathfrak{g} = \dim \mathfrak{h}$. We denote by $R = R(\mathfrak{g}, \mathfrak{h}) = R([\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{h}) \subset \mathfrak{h}^*$ the associated *root system*. Recall that the *Weyl group* $W = W(\mathfrak{g}, \mathfrak{h})$ of R can be naturally identified with $N_G(\mathfrak{h})/Z_G(\mathfrak{h}) \subset \mathrm{GL}(\mathfrak{h})$ (see, for example, [TY, 30.6.5]). The type of the root system R, as well as the type of the reflection group W, will be indicated by capital roman letters, frequently indexed by the rank of $[\mathfrak{g}, \mathfrak{g}]$, e.g. E_8. If $\alpha \in R(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ is the *root subspace* associated to α. If M is a subset of $R(\mathfrak{g}, \mathfrak{h})$, we denote by $\langle M \rangle$ the root subsystem $(\sum_{\alpha \in M} \mathbb{Q}\alpha) \cap R(\mathfrak{g}, \mathfrak{h})$.

We use the notation $\lfloor\ \rfloor$, resp. $\lceil\ \rceil$, for the floor, resp. ceiling, function on $\mathbb{Q}$; thus $\lfloor\lambda\rfloor$, resp. $\lceil\lambda\rceil$, is the largest, resp. smallest, integer $\leq \lambda$, resp. $\geq \lambda$.

1.2 Levi factors

We start by recalling the definition of Levi factors:

Definition 1.2.1. A *Levi factor* of $\mathfrak{g}$ is a subalgebra of the form $\mathfrak{l} = \mathfrak{g}^s$ where $s \in \mathfrak{g}$ is semisimple. The connected algebraic subgroup $L \subset G$ associated to the (algebraic) subalgebra $\mathfrak{l}$ is called a *Levi factor* of G.

Observe that the previous definition of a Levi factor of $\mathfrak{g}$ is equivalent to the definition given in [TY, 29.5.6], see, for example, [Bou, Exercice 10, p. 223]. Recall that a Levi factor $\mathfrak{l} = \mathfrak{g}^s$ is reductive [TY, 20.5.13] and $L = G^s$, cf. [St, Corollary 3.11] and [TY, 24.3.6].

Let $\mathfrak{h}$ be a Cartan subalgebra and $\mathfrak{l}$ be a Levi factor containing $\mathfrak{h}$. By [TY, 20.8.6] there exists a subset $M = M_{\mathfrak{l}} \subset R(\mathfrak{g}, \mathfrak{h})$ such that $M = \langle M\rangle$ and

$$\mathfrak{l} = \mathfrak{l}_M = \mathfrak{h} \oplus \bigoplus_{\alpha\in M} \mathfrak{g}^{\alpha} \tag{1.2}$$

$$\mathfrak{c}_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) = \{t \in \mathfrak{h} \mid \alpha(t) = 0 \text{ for all } \alpha \in M\} \text{ and } \mathfrak{c}_{\mathfrak{g}}(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{l})) = \mathfrak{l}. \tag{1.3}$$

Conversely, if $M \subset R(\mathfrak{g}, \mathfrak{h})$ is a subset such that $M = \langle M\rangle$, define $\mathfrak{l} = \mathfrak{l}_M$ as in equation (1.2); then $\mathfrak{l}_M$ is a Levi factor and:

$$\mathfrak{h} \supseteq \{s \in \mathfrak{g} \mid \mathfrak{l}_M = \mathfrak{g}^s\} = \ker M \setminus \left(\bigcup_{\alpha\notin M} \ker\alpha\right) \neq \emptyset. \tag{1.4}$$

This construction gives a bijective correspondence $\mathfrak{l} = \mathfrak{l}_M \leftrightarrow M = M_{\mathfrak{l}}$ between Levi factors and subsets of $R(\mathfrak{g}, \mathfrak{h})$ satisfying the above property. Remark that the Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ acts on the set of Levi factors by its action on $R(\mathfrak{g}, \mathfrak{h})$. Precisely, if $g \in N_G(\mathfrak{h})$ and $\mathfrak{l} \supset \mathfrak{h}$ is a Levi factor, one has $g.\mathfrak{l} = w.\mathfrak{l}$ where $w = gZ_G(\mathfrak{h}) \in W$ is the class of g. Let $x, y \in \mathfrak{h}$; we will say that the Levi factors $\mathfrak{g}^x, \mathfrak{g}^y$ are W-conjugate if there exists $w \in W$ such that $w.M_{\mathfrak{g}^x} = M_{\mathfrak{g}^y}$. From (1.4) one deduces that this definition is equivalent to $w.\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^x) = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^y)$ for some $w \in W$.

Assume that $\mathfrak{g}$ is semisimple and denote by κ the isomorphism $\mathfrak{h} \xrightarrow[\kappa]{\sim} \mathfrak{h}^*$ induced by the restriction of the Killing form of $\mathfrak{g}$. Define a $\mathbb{Q}$-form of $\mathfrak{h}$, or $\mathfrak{h}^*$, by $\mathfrak{h}_{\mathbb{Q}} \overset{\kappa}{\cong} \mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q}.R(\mathfrak{g}, \mathfrak{h})$. Fix the Cartan subalgebra $\mathfrak{h}$ and a fundamental system (i.e. a basis) B of $R(\mathfrak{g}, \mathfrak{h})$. We say that a Levi factor $\mathfrak{l}$ is *standard* if $\mathfrak{l} = \mathfrak{g}^s$ with $s \in \mathfrak{h}_{\mathbb{Q}}$ in the positive Weyl chamber of associated to B. In this case, one can write $M_{\mathfrak{l}} = \langle I_{\mathfrak{l}}\rangle = \mathbb{Z}I_{\mathfrak{l}} \cap R(\mathfrak{g}, \mathfrak{h})$ where $I_{\mathfrak{l}} \subset B$. The following proposition is consequence of the definition of a Levi factor and (1.4).

Proposition 1.2.2. *Any Levi factor of $\mathfrak{g}$ is G-conjugate to a standard Levi factor.*

Let $\mathfrak{l} \subset \mathfrak{g}$ be a Levi factor and L be the associated Levi factor of G. There exists a unique decomposition $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \bigoplus_i \mathfrak{l}_i$, where $\mathfrak{z}(\mathfrak{l})$ is the centre and the $\mathfrak{l}_i$ are simple subalgebras. Let

$L_i \subset G$ be the connected subgroup with Lie algebra $\mathfrak{l}_i$ (cf. [TY, 24.7.2]). Under this notation we have:

Proposition 1.2.3. *The subgroup $L \subset G$ is generated by $C_G(\mathfrak{l})$ and the subgroups L_i.*

Proof. Recall that $\operatorname{Lie} L_i = \mathfrak{l}_i$ and $\operatorname{Lie} Z_G(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l})$. By [TY, 24.5.9] one gets that L is generated by the connected subgroups L_i and $C_G(\mathfrak{l})^\circ$. Writing $\mathfrak{l} = \mathfrak{g}^s$ with s semisimple, we have already observed that $L = G^s$, hence $C_G(\mathfrak{g}^s) \subset G^s$ and the result follows. $\square$

1.3 Jordan G-classes

The description of G-sheets is closely related to the study of Jordan G-classes, also called decomposition classes. We now recall some facts about these classes (see, for example, [BK, Boh, Bro, TY]).

Recall from §1.1 that any element $x \in \mathfrak{g}$ has a unique Jordan decomposition $x = s + n$. We then say that the pair $(\mathfrak{g}^s, n)$ is the *datum* of x.

Definition 1.3.1. Let $x = s + n$ be the Jordan decomposition of $x \in \mathfrak{g}$. The Jordan G-class of x, or J_G-class of x, is the set $J_G(x) := G.(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)^\bullet + n)$. Two elements are Jordan G-equivalent if they have the same J_G-class.

Let L be a Levi factor of G with Lie algebra $\mathfrak{l}$, and $L.n \subset \mathfrak{l}$ be a nilpotent orbit. If J is a J_G-class, the pair $(\mathfrak{l}, L.n)$, or $(\mathfrak{l}, n)$, is called a *datum of* J if $(\mathfrak{l}, n)$ is the datum of an element $x \in J$. Setting $\mathfrak{t} := \mathfrak{g}^{\mathfrak{l}}$ it is then easy to see that $J = G.(\mathfrak{t}^\bullet + n)$. From this result one can deduce that Jordan G-classes are locally closed [TY, 39.1.7], and smooth [Bro]. Furthermore, two elements of $\mathfrak{g}$ are Jordan G-equivalent if and only if their data are conjugate under the diagonal action of G [TY, 39.1]. Then, $\mathfrak{g}$ is the finite disjoint union of its Jordan G-classes (cf. [TY, 39.1.8]). Since each G-Jordan class is an irreducible subvariety of some $\mathfrak{g}^{(m)}$, we get the following result:

Proposition 1.3.2. *A G-sheet of $\mathfrak{g}$ is a finite (disjoint) union of Jordan G-classes.*

An immediate consequence of this proposition is that each G-sheet S contains a unique dense (open) Jordan G-class J. It follows that we can define the *datum of* S to be any datum $(\mathfrak{l}, L.n)$, or $(\mathfrak{l}, n)$, of this dense class J. For instance, if S is a G-sheet containing a semisimple element, i.e. S is a *Dixmier sheet*, then J is the class of semisimple elements of S and $(\mathfrak{l}, 0)$ is a datum of S, see [TY, 39.4.5].

1.4 Slodowy slices

We recall in this subsection some of the important results obtained by Katsylo [Kat]. One of the first fundamental properties of the sheets in $\mathfrak{g}$ was obtained by Borho-Kraft [BK, Korollar 5.8] (cf. also [TY, 39.3.5]):

Proposition 1.4.1. *Each G-sheet contains a unique nilpotent orbit.*

Fix a G-sheet S_G, a datum $(\mathfrak{l}, L.n)$ of S_G, cf. 1.3, and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{l}$. Set $\mathfrak{t} := \mathfrak{g}^{\mathfrak{l}}$ (thus $\mathfrak{t} \subset \mathfrak{h}$). Then, following [BK], one can construct a parabolic subalgebra $\mathfrak{j}$ of $\mathfrak{g}$ and a nilpotent ideal $\mathfrak{n}$ of $\mathfrak{j}$ such that $\mathfrak{r} = \mathfrak{n} \oplus \mathfrak{t}$ satisfies $S_G = G.\mathfrak{r}^\bullet$ (and $\overline{S_G} = G.\mathfrak{r}$). This is done as follows. Recall, see for example [Ca, §5.7], that there exists a grading $\mathfrak{l} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{l}_i$ such that $\mathfrak{j}_2 = \bigoplus_{i\geq 0} \mathfrak{l}_i$ is a parabolic subalgebra of $\mathfrak{l}$, $\mathfrak{n}_2 = \bigoplus_{i\geq 2} \mathfrak{l}_i$ is a nilpotent ideal of $\mathfrak{j}_2$ such that $[\mathfrak{j}_2, n] = \mathfrak{n}_2$. If $\mathfrak{n}_1$ is the nilradical of any parabolic subalgebra with $\mathfrak{l}$ as Levi factor, one then takes $\mathfrak{j} = \mathfrak{j}_2 + \mathfrak{n}_1$ and $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$.

Note here that when S_G is *Dixmier*, *i.e.* contains semisimple elements, then $n = 0$ and $\mathfrak{j} = \mathfrak{l} + \mathfrak{n}$ has $\mathfrak{l}$ as Levi factor and $\mathfrak{n}$ as nilradical. This will be the case when S_G is regular in section 1.5 or when $\mathfrak{g}$ is of type A in 1.6.

Under the previous notation, the following result is proved in [Kat, Lemma 3.2] (cf. also [IH, Proposition 2.6]).

Proposition 1.4.2. *Let (e, h, f) be an $\mathfrak{sl}_2$-triple such that $e \in \mathfrak{n}^\bullet$ and $h \in \mathfrak{h}$, then*

$$S_G = G.(e + \mathfrak{t}).$$

From [Kat, Lemma 3.1] one knows that there exists an $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, h, f)$ such that $e \in \mathfrak{n}^\bullet$ and $h \in \mathfrak{h}$. We fix $\mathscr{S} = (e, h, f)$ for the rest of the subsection. Note that $e \in S_G$. The adjoint action of h on $\mathfrak{g}$ yields a gradation

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i, h), \quad \mathfrak{g}(i, h) = \{v \in \mathfrak{g} : [h, v] = iv\}.$$

One of the main constructions in [Kat] consists in deforming the "section" $e + \mathfrak{t}$ into an other "section" having nice properties. The construction goes as follows. First, define a subset $X(S_G, \mathscr{S}) \subset S_G$, depending only on the sheet and the choice of the $\mathfrak{sl}_2$-triple, by:

$$e + X(S_G, \mathscr{S}) := S_G \cap (e + \mathfrak{g}^f).$$

Then, the deformation is made by using a map $\varepsilon^{\mathfrak{g}}_{S_G,\mathscr{S}} : e + \mathfrak{t} \to e + X(S_G, \mathscr{S})$, whose definition is recalled below, see Remark 1.4.5. Before going into the details, note that when there is no ambiguity on the context, we write X instead of $X(S_G, \mathscr{S})$ and $\varepsilon^{\mathfrak{g}}$, or ε, instead of $\varepsilon^{\mathfrak{g}}_{S_G,\mathscr{S}}$.

Remark 1.4.3. When $\mathfrak{g}$ is of type A, there is a unique sheet containing a fixed nilpotent orbit (cf. [Kr, §2]). In this case we can therefore set $X(\mathscr{S}) := X(S_G, \mathscr{S})$ where S_G is the sheet containing the nilpotent element e of $\mathscr{S}$.

Define a one parameter subgroup $(F_t)_{t\in\Bbbk^\times} \subset \mathrm{GL}(\mathfrak{g})$ by setting $F_t.y = t^{(i-2)}y$ for $y \in \mathfrak{g}(i, h)$. One can show that $F_t.e = e$, $F_t.S_G = S_G$, $F_t.X = X$ and $\lim_{t\to 0} F_t.y = e$ for all $y \in e + X$. One can slightly modify [Kat, Lemma 5.1] to obtain the following result:

Lemma 1.4.4. *There exists a polynomial map*

$$\epsilon : e + \bigoplus_{i \leqslant 0} \mathfrak{g}(2i,h) \longrightarrow e + \mathfrak{g}^f$$

such that:

(i) $e + z \in G.\epsilon(e+z)$ *for all* $z \in \bigoplus_{i\leqslant 0} \mathfrak{g}(2i,h)$*;*

(ii) *let* $j \leqslant 0$ *and set* $P_j = (\pi_{2j} \circ \epsilon)_{|e+\mathfrak{g}(0,h)}$ *where* π_{2j} *is the canonical projection from* $\bigoplus_{i\leqslant 0} \mathfrak{g}(2i,h)$ *onto* $\mathfrak{g}(2j,h)$*, then* P_j *is either* 0 *or a homogeneous polynomial of degree* $-j+1$*.*

Proof. We set $\mathfrak{g}_i = \mathfrak{g}(i,h)$ for $i \leqslant 1$. One can then define affine subspaces L_{2i} and M_{2i} by:

$$L_{2i} := \mathfrak{g}^f \cap \mathfrak{g}_{2i}, \quad M_{2i} := e + L_2 + L_0 + L_{-2} + \dots + L_{2i} + \mathfrak{g}_{2i-2} + \mathfrak{g}_{2i-4} + \cdots$$

It is clear that $L_2 = \{0\}$, $M_2 = e + \bigoplus_{i\leqslant 0} \mathfrak{g}_{2i}$ and $M_{-2k} = e + \mathfrak{g}^f$ for k large enough. We fix such a k. Now, define maps $\epsilon_i : M_{2i} \to M_{2i-2}$ as follows.
Denote the projections associated to the decomposition $\mathfrak{g}_{2i-2} = [e, \mathfrak{g}_{2i-4}] \oplus L_{2i-2}$ by $\mathrm{pr}_1 : \mathfrak{g}_{2i-2} \to [e, \mathfrak{g}_{2i-4}]$ and $\mathrm{pr}_2 : \mathfrak{g}_{2i-2} \to L_{2i-2}$ (hence $\mathrm{pr}_1 + \mathrm{pr}_2 = \mathrm{Id}_{\mathfrak{g}_{2i-2}}$). Next, define $\eta_{2i-2} : \mathfrak{g}_{2i-2} \to \mathfrak{g}_{2i-4}$ to be the linear map $(\mathrm{ad}\, e)^{-1} \circ \mathrm{pr}_1$, which satisfies $[\eta_{2i-2}(x), e] + x \in L_{2i-2}$ for all $x \in \mathfrak{g}_{2i-2}$. If $e + z = e + \sum_{j=i}^{0} z_{2j} + \sum_{j=k}^{i-1} w_{2j} \in M_{2i}$, where $z_{2j} \in L_{2j}$, $w_{2j} \in \mathfrak{g}_{2j}$, set:

$$\epsilon_i(e+z) = \exp(\mathrm{ad}\, \eta_{2i-2}(w_{2i-2}))(e+z).$$

Then, ϵ_i is a polynomial map such that $\epsilon_i(e+z) \in M_{2i-2}$. Now, set:

$$\epsilon'_i = \epsilon_i \circ \dots \circ \epsilon_{-1} \circ \epsilon_0 \circ \epsilon_1, \quad \epsilon = \epsilon'_{-k}.$$

Clearly, ϵ is a polynomial map which satisfies (i).

To get (ii), we now show, by decreasing induction on $i \leqslant 2$, that $(\pi_{2j} \circ \epsilon'_i)_{|e+\mathfrak{g}_0}$ is either 0 or a homogeneous polynomial of degree $-j+1$. Set $\epsilon'_2 = \mathrm{Id}$, for which the claim is obviously true. Assume that the assertion is true for a given integer $i_0 = i+1 \leqslant 2$. Remark that the construction of ϵ_i, ϵ'_i gives

$$\epsilon'_i(e+t) = \epsilon_i \circ \epsilon'_{i_0}(e+t) = \exp(\mathrm{ad}\, \eta_{2i-2}(\pi_{2i-2} \circ \epsilon'_{i_0}(e+t))).\epsilon'_{i_0}(e+t)$$

for all $e + t \in e + \mathfrak{g}_0$. By induction, $u_i := \eta_{2i-2}(\pi_{2i-2} \circ \epsilon'_{i_0}) : e + \mathfrak{g}_0 \to \mathfrak{g}_{2i-4}$ is 0 or homogeneous of degree $-i+2$; thus

$$\pi_{2j} \circ \epsilon'_i(e+t) = \sum_{l \geqslant 0} \frac{(\mathrm{ad}\, u_i(e+t))^l}{l!} \circ \pi_{2j+l(-2i+4)} \circ \epsilon'_{i+1}(e+t)$$

is either 0 or homogeneous of degree $l(-i+2) + (-j - l(-i+2) + 1) = -j+1$, as desired. $\square$

Remark 1.4.5. The polynomial map ϵ constructed in the proof of Lemma 1.4.4 will be denoted by $\epsilon^{\mathfrak{g}} = \epsilon^{\mathfrak{g}}_{\mathscr{S}}$. By restriction, it induces a map $\varepsilon = \varepsilon^{\mathfrak{g}} = \varepsilon^{\mathfrak{g}}_{\mathscr{S}}$ from $e+\mathfrak{h}$ to $e+\mathfrak{g}^f$ and Lemma 1.4.4(i) implies that ϵ maps $e+\mathfrak{t}$ into $e+X$. One can therefore define $\varepsilon^{\mathfrak{g}}_{S_G,\mathscr{S}}$ to be the polynomial map $(\varepsilon^{\mathfrak{g}}_{\mathscr{S}})_{|e+\mathfrak{t}}$.
Furthermore, one may observe that the construction of $\epsilon^{\mathfrak{g}}$ made in the proof of the previous proposition yields that $\varepsilon^{\mathfrak{g}}$ does not depend on $\mathfrak{g}$ in the following sense: if $\mathfrak{g}'$ is a reductive Lie subalgebra of $\mathfrak{g}$ containing $\mathscr{S}$, then $\varepsilon^{\mathfrak{g}'} = \varepsilon^{\mathfrak{g}}_{|\mathfrak{h}\cap\mathfrak{g}'}$. In the sequel, we will often write ε when the subscript is obvious from the context.

The next lemma is due to Katsylo [Kat], see [IH] for a purely algebraic proof.

Lemma 1.4.6. *Under the previous notation:*
(i) $S_G = G.(e+X)$ *;*
(ii) *the action of G on $\mathfrak{g}$ induces an action of $A = G^e/(G^e)^\circ$ on $e+X$;*
(iii) *for all $x \in e+X$, one has $A.x = G.x \cap (e+X)$.*

These results enable us to define a quotient map (of sets) by:

$$\psi = \psi_{S_G,\mathscr{S}} : S_G \longrightarrow (e+X)/A, \quad \psi(x) = A.y \ \text{ if } G.y = G.x.$$

Since $e+X$ is an affine algebraic variety on which the finite group A acts rationally, it follows from [TY, 25.5.2] that $(e+X)/A$ can be endowed (in a canonical way) with a structure of algebraic variety and that the quotient map

$$\gamma : e+X \longrightarrow (e+X)/A \tag{1.5}$$

is the geometric quotient of $e+X$ under the action of A. Using Lemma 1.4.4(i) and Lemma 1.4.6 one obtains:

$$\psi = \gamma \circ \varepsilon \ \text{ on } e+\mathfrak{t}.$$

The following theorem is the main result in [Kat]:

Theorem 1.4.7. *The map $\psi : S_G \to (e+X)/A$ is a morphism of algebraic varieties and gives a geometric quotient S_G/G of the sheet S_G.*

Remark 1.4.8. One has $\dim S_G/G = \dim X = \dim \mathfrak{t}$, see [Boh, §5]. It is shown in [IH, Corollary 4.6] that, when $\mathfrak{g}$ is classical, the map $\varepsilon : e+\mathfrak{t} \to e+X$ is quasi-finite (it is actually finite by [IH, Chaps. 5 & 6]).

The variety $e+X$ will be called a *Slodowy slice* of S_G. One of the main results of [IH] is that $e+X$ is smooth when $\mathfrak{g}$ is of classical type, cf. Theorem 1.4.10. This result relies on some properties of $e+\mathfrak{g}^f$ that we now recall (see [Sl, 7.4]).

Proposition 1.4.9. (i) *The intersection of $G.x$ with $e+\mathfrak{g}^f$ is transverse for any $x \in e+X$ (*i.e. *$T_x(G.x) \cap T_x(e+\mathfrak{g}^f) = \{x\}$ and $T_x(e+\mathfrak{g}^f) + T_x(G.x) = T_x(\mathfrak{g})$.)*

(ii) *The morphism* $\delta : G \times (e + \mathfrak{g}^f) \to \mathfrak{g}$, $(g, x) \mapsto g.x$, *is smooth.*
(iii) *Let* Y *be a* G*-stable subvariety of* $\mathfrak{g}$ *and set* $Z = Y \cap (e + \mathfrak{g}^f)$*. Then the restricted morphism* $\delta' : G \times Z \to Y$ *is smooth. In particular, when* $Y = G.Z$*,* Y *is smooth if and only if* Z *is smooth.*

Proof. Claims (i) and (ii) are essentially contained in [Sl, 7.4, Corollary 1].
(iii) We merely repeat the argument given in [IH]. Let $\hat{Z} = Y \cap_{\mathrm{sch}} (e + \mathfrak{g}^f) = Y \times_{\mathfrak{g}} (e + \mathfrak{g}^f)$ be the schematic intersection of Y and $(e + \mathfrak{g}^f)$ (cf. [Har, p. 87]). Writing $(G \times (e + \mathfrak{g}^f)) \times_{\mathfrak{g}} Y \cong G \times ((e + \mathfrak{g}^f) \times_{\mathfrak{g}} Y) = G \times \hat{Z}$, the base extension $Y \to \mathfrak{g}$ gives the following diagram:

$$\begin{array}{ccc} G \times (e + \mathfrak{g}^f) & \xrightarrow{\delta} & \mathfrak{g} \\ \uparrow & & \cup \\ G \times \hat{Z} & \xrightarrow{\delta''} & Y. \end{array}$$

By [Har, III, Theorem 10.1] δ'' is smooth. Thus, as Y is reduced, [AK, VII, Theorem 4.9] implies that $\hat{Z}$ is reduced. Since Y is G-stable, it is easy to see that δ' factorizes through δ'', hence $\delta' = \delta''$. When $Y = G.Z$, the morphism δ' is surjective and [AK, VII, Theorem 4.9] then implies that Z is smooth if, and only if, Y is smooth. □

Applying Proposition 1.4.9(iii) to a sheet $Y = S_G$, one deduces that S_G is smooth if and only if the Slodowy slice $e + X$ is smooth. Using this method, the following general result was obtained by Im Hof:

Theorem 1.4.10 ([IH])**.** *The sheets of a classical Lie algebra are smooth.*

Recall that the smoothness of sheets for $\mathfrak{sl}_N$ is due to Kraft and Luna [Kr] and, independentely, Peterson [Pe]. It is known that when $\mathfrak{g}$ is of type G_2, a subregular sheet of $\mathfrak{g}$ is not normal (hence is singular), see [Sl, 8.11], [Boh, 6.4] or [Pe]. It seems to be the only known example of non smooothness of sheets.

Remarks 1.4.11. (1) Let $\mathscr{S} = (e, h, f)$ be as above and pick $g \in G$. Then, the same results can be obtained for $g.e$ and $g.\mathscr{S}$. In particular, one can construct a map

$$\varepsilon : g.e + g.\mathfrak{h} \to g.e + \mathfrak{g}^{g.f}$$

which induces a polynomial map $\varepsilon_{|g.e+g.\mathfrak{t}}$.
(2) The results obtained from 1.4.6 to 1.4.10 depend only on S_G and $\mathscr{S}$ but do not refer to $\mathfrak{l}$, $\mathfrak{t}$ or $\mathfrak{n}$. Precisely, these results remain true when e is replaced by $g.e$ and $\mathscr{S}$ by any $\mathfrak{sl}_2$-triple containing $g.e$. In particular, since S_G contains a unique nilpotent G-orbit $G.e$, they remain true for any $\mathfrak{sl}_2$-triple (e', h', f') such that $e' \in S_G$.

1.5 The regular G-sheet

The set $\mathfrak{g}^{reg}$ of regular elements in $\mathfrak{g}$ is a sheet, called the regular G-sheet, that we will denote by S_G^{reg}. We will use the notation and results of the previous subsection with $S_G = S_G^{reg}$. One has $\mathfrak{t} = \mathfrak{h}$ and $G.(e + \mathfrak{h}) = S_G^{reg}$ for any principal $\mathfrak{sl}_2$-triple (e, h, f) such that e is regular and $h \in \mathfrak{h}$. Moreover, $e + \mathfrak{g}^f \subset S_G^{reg}$ and therefore $S_G^{reg} = G.(e + \mathfrak{g}^f)$ (cf. [Ko]).

Lemma 1.5.1. *Adopt the previous notation.*
(i) *The semisimple part of an element $e + x \in e + \mathfrak{h}$ is conjugate to x.*
(ii) *Two regular elements are conjugate if and only if their semisimple parts are in the same G-orbit.*
(iii) *Two elements $e + x, e + y \in e + \mathfrak{h}$ lie in the same G-orbit if and only if $W.x = W.y$.*

Proof. The assertions (i) and (ii) follow from [Ko, Lemma 11, Theorem 3], whence (iii) is a direct consequence of (i) and (ii). □

We can now state an important result of Kostant [Ko, Theorem 8] under the following form:

Lemma 1.5.2. *The group A is trivial, thus $\psi : S_G^{reg} \to e + \mathfrak{g}^f = \varepsilon(e + \mathfrak{h})$ is a geometric quotient of S_G^{reg}.*

1.6 The case $\mathfrak{g} = \mathfrak{gl}_N$

1.6.1 The setting

In this section we assume that $\mathfrak{g} = \mathfrak{gl}(V)$, where V is a $\Bbbk$-vector space of dimension N. By [Kr, §2], we know that there exist two natural bijections from G-sheets to partitions of N:

- the first one, associates to a G-sheet S the partition $\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\delta_{\mathcal{O}}})$ of the unique nilpotent orbit $\mathcal{O}$ contained in S (cf. Proposition 1.4.1);
- the second one, sends a G-sheet S to the partition $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}_1 \geqslant \cdots \geqslant \tilde{\lambda}_{\delta_{\mathfrak{l}}})$ given by the block sizes of the Levi factor $\mathfrak{l}$ occurring in the datum $(\mathfrak{l}, 0)$ of the dense J_G-class contained in S.

These two bijections are related by the fact that $\tilde{\boldsymbol{\lambda}}$ is the transpose of $\boldsymbol{\lambda}$.

Let S_G be a G-sheet and $\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\delta_{\mathcal{O}}})$ be the partition of N associated to the nilpotent orbit $\mathcal{O}$ contained in S_G. Fix an element $e \in \mathcal{O}$ and a basis

$$\mathbf{v} = \{v_j^{(i)} \mid i \in [\![1, \delta_{\mathcal{O}}]\!], j \in [\![1, \lambda_i]\!]\}$$

providing a Jordan normal form of e. Precisely, write $e = \sum_i e_i$, where $e_i \in \mathfrak{g}$ is defined by:

$$e_i.v_j^{(k)} = \begin{cases} v_{j-1}^{(i)} & \text{if } k = i \text{ and } j = 2, \ldots, \lambda_i; \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

Set $\mathfrak{q}_i = \mathfrak{gl}(v_j^{(i)} \mid j \in [\![1, \lambda_i]\!])$, which is a reductive Lie algebra isomorphic to $\mathfrak{gl}_{\lambda_i}$, and define

$$\mathfrak{q} := \bigoplus_i \mathfrak{q}_i.$$

Let $\mathrm{pr}_i : \mathfrak{q} \to \mathfrak{q}_i$ be canonical projection. For $x \in \mathfrak{q}$ we set $x_i = \mathrm{pr}_i(x)$; conversely, for any family $(y_i)_i$ of elements $y_i \in \mathfrak{q}_i$ we can define $y = \sum_i y_i \in \mathfrak{q}$.
We apply this construction to get an $\mathfrak{sl}_2$-triple $(e, h, f) \subset \mathfrak{q}$ as follows. Fixing the basis $(v_1^{(i)}, \dots, v_{\lambda_i}^{(i)})$, one can identify $\mathfrak{q}_i$ with the algebra of $\lambda_i \times \lambda_i$-matrices. Using this identification, embed e_i in the standard $\mathfrak{sl}_2$-triple (e_i, h_i, f_i) of $\mathfrak{q}_i$ afforded by the irreducible representation of $\mathfrak{sl}_2$ of dimension λ_i, i.e.:

$$e_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad h_i = \begin{pmatrix} \lambda_i - 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i - 3 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i - 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_i + 1 \end{pmatrix}$$

(a well known similar formula gives f_i). Then, $h = \sum_i h_i$ and $f = \sum_i f_i$.
Clearly, the subspace

$$\mathfrak{l} = \bigoplus_j \mathfrak{gl}(v_j^{(i)} \mid i \in [\![1, \tilde{\lambda}_j]\!])$$

is a Levi factor of $\mathfrak{g}$. Denote by $\mathfrak{h} = \bigoplus_i \mathfrak{h}_i$ the Cartan subalgebra of diagonal matrices with respect to the chosen basis $\mathbf{v}$. If $\mathfrak{t}$ is the center of $\mathfrak{l}$ we then have $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{l} \cap \mathfrak{q}$.
Let $E_{j,j}^{i,i}$ be the element of $\mathfrak{h}$ defined by $E_{j,j}^{i,i}.v_k^{(l)} = v_j^{(i)}$ if $(i,j) = (l,k)$, and $E_{j,j}^{i,i}.v_k^{(l)} = 0$ otherwise. Each $t \in \mathfrak{h}$ can then be written $t = \sum_{i,j} t_{i,j} E_{j,j}^{i,i}$ and one has the following easy characterization of $\mathfrak{t}$:

$$\mathfrak{t} = \{t \in \mathfrak{h} \mid t_{i,j} = t_{i',j} \text{ for all } i \leqslant i';\ j \in [\![1, \lambda_{i'}]\!]\}. \tag{1.7}$$

We will need later the following isomorphism:

$$\alpha : \left\{ \begin{array}{ccc} \mathbb{k}^{\lambda_1} & \xrightarrow{\sim} & \mathfrak{t} \\ (x_j)_{j \in [\![1,\lambda_1]\!]} & \mapsto & (t_{i,j})_{i,j} \end{array} \right. \tag{1.8}$$

where $t_{i,j} = x_j$ for all $i \in [\![1, \lambda_{\delta_{\mathcal{O}}}]\!]$, $1 \leqslant j \leqslant \lambda_i$.

Order, lexicographically, the elements of $\mathbf{v}$ by: $v_j^{(i)} < v_\ell^{(k)}$ if $j < \ell$ or $j = \ell$ and $i < k$. Denote by $\mathfrak{b}$ the Borel subalgebra of $\mathfrak{g}$ consisting of upper triangular matrices with respect to this ordering of $\mathbf{v}$. Then, the subspace $\mathfrak{b} + \mathfrak{l}$ is a parabolic subalgebra having $\mathfrak{l}$ as Levi factor. Observe that $h \in \mathfrak{h} \subset \mathfrak{l}$ and that e is regular in the nilradical of $\mathfrak{b} + \mathfrak{l}$. The constructions of §1.4 can be made here with $\mathfrak{j} = \mathfrak{b} + \mathfrak{l}$ and the results of that subsection yield $S_G = G.(e + \mathfrak{t})$ (Proposition 1.4.2) and a map $\varepsilon : e + \mathfrak{h} \to e + \mathfrak{g}^f$ (Lemma 1.4.4).

Lemma 1.6.1. (i) *The group G^e is connected.*
(ii) *The map ψ induces a bijection between G-orbits in S_G and points in X.*

Proof. Part (i) is a classical result, see for example [CM, 6.1.6]. Since the group $A = G^e/(G^e)^\circ$ is then trivial, part (ii) follows from Lemma 1.4.6. $\square$

By Remark 1.4.5 we may assume that $\varepsilon = \varepsilon^{\mathfrak{q}} = \sum_i \varepsilon_i$ where

$$\varepsilon_i := \varepsilon^{\mathfrak{q}_i} : e_i + \mathfrak{h}_i \to e_i + \mathfrak{q}_i^{f_i}.$$

As $e_i \in \mathfrak{q}_i$ is regular, the study of ε is therefore reduced to the regular case.

1.6.2 The regular case and its consequences

We need to study in more details the maps $\varepsilon_i : e_i + \mathfrak{h}_i \to e_i + \mathfrak{q}_i^{f_i}$ introduced at the end of the previous subsection, where, as already said, e_i is regular in $\mathfrak{q}_i \cong \mathfrak{gl}_{\lambda_i}$.
To simplify the notation we (temporarily) replace $\mathfrak{gl}_{\lambda_i}$ by $\mathfrak{gl}_N$ and e_i by e^{reg}, the regular element of $\mathfrak{g} = \mathfrak{gl}_N$. Hence,

$$e^{reg}.v_j = \begin{cases} v_{j-1} & \text{if } j = 2, \dots, N; \\ 0 & \text{if } j = 1. \end{cases}$$

Recall that $\mathfrak{h} \subset \mathfrak{gl}_N$ is the set of diagonal matrices in the basis $\mathbf{v}^{reg} = (v_j)_j$. We can then define the canonical principal triple $(e^{reg}, h^{reg}, f^{reg})$ with respect to this basis (see the definition of the triple (e_i, h_i, f_i) in 1.6.1). In this case, $\varepsilon^{reg} : e^{reg} + \mathfrak{h} \to e^{reg} + \mathfrak{g}^{f^{reg}}$ can be considered as the restriction of the geometric quotient map of $\mathfrak{g}^{reg}$ (cf. Lemma 1.5.2).

Let $0 \leqslant k < N$, the *k-th subdiagonal* (resp. k-th *supdiagonal*) is the subspace of matrices $[a_{i,j}]_{i,j}$ such that $a_{i,j} = 0$ unless $i = j + k$ (resp. $i = j - k$). We denote it by $\mathfrak{f}^{(k)}$.

Lemma 1.6.2. *The map ε^{reg} is given by*

$$\varepsilon^{reg}(e^{reg} + t) = e_i + \sum_{j \leqslant 0} P_j(t) \quad \textit{for all } t \in \mathfrak{h},$$

where each $P_j : \mathfrak{h} \to \mathfrak{f}^{(-j)}$ is a homogeneous polynomial map of degree $-j+1$, symmetric in the eigenvalues of the elements of $\mathfrak{h}$.

Proof. Recall that $\mathfrak{g}(2j, h^{reg})$ is the $2j$-th eigenspace of $\mathrm{ad}_{\mathfrak{g}}\, h^{reg}$. It is easily seen that $\mathfrak{g}(2j, h^{reg}) = \mathfrak{f}^{(-j)}$ when $j \leqslant 0$. Using Lemma 1.4.4(ii), the only fact remaining to be proved is that the polynomial map P_j is symmetric. Observe that the Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ acts as the permutation group of $[\![1, N]\!]$ on the eigenvalues of $\mathfrak{h}$ and recall that, by Lemma 1.5.2, ε^{reg} is a quotient map with respect to W. Consequently, for all $t \in \mathfrak{h}$ and $w \in W$ one has $\varepsilon^{reg}(e^{reg} + w.t) = \varepsilon^{reg}(e^{reg} + t)$. Thus P_j is symmetric. $\square$

If t is a semisimple element of $\mathfrak{g}$ we denote by $\mathsf{sp}(t)$ the set of eigenvalues of t and by $m(t, c)$ the multiplicity of $c \in \Bbbk$ as an eigenvalue of t, with the convention that $m(t, c) = 0$ if $c \notin \mathsf{sp}(t)$. The next lemma is a direct consequence of Lemma 1.5.1.

Lemma 1.6.3. *Let $t \in \mathfrak{h}$ and $c \in \mathsf{sp}(t)$. In a Jordan normal form of $e^{reg} + t$, there exists exactly one Jordan block associated to c, and its size is $m(t, c)$.*

Recall that we want to apply Lemma 1.6.3 to the regular elements e_i in $\mathfrak{q}_i \cong \mathfrak{gl}_{\lambda_i}$; we therefore generalize the previous notation as follows. For $t = \sum_i t_i \in \mathfrak{h} \subset \bigoplus_i \mathfrak{q}_i$ and $c \in \Bbbk$, let $m_i(t, c)$ be the multiplicity of c as an eigenvalue of t_i. Then, $\sum_i m_i(t, c) = m(t, c)$ and we have the following easy consequence of Lemmas 1.5.1 and 1.6.3.

Corollary 1.6.4. *Let $t \in \mathfrak{h}$. The semisimple part of $e + t$ is conjugate to t. Its nilpotent part is associated to the partition of N given by the integers $m_i(t, c)$, $c \in \mathsf{sp}(t)$ and $i \in [\![1, \delta_{\mathcal{O}}]\!]$.*

1.7 Reduction to simple Lie algebras

Let $\mathfrak{g} = \prod_i \mathfrak{g}_i = \bigoplus_i \mathfrak{g}_i$ be a decomposition of $\mathfrak{g}$ as a direct sum of reductive Lie (sub)algebras. Let G_i be the adjoint group of $\mathfrak{g}_i$, thus $G = \prod_i G_i$.

Lemma 1.7.1. *The G-sheets of $\mathfrak{g}$ are of the form $\prod_i S_i$ where each S_i is a G_i-sheet of $\mathfrak{g}_i$.*

Proof. Clearly, an obvious induction reduces the proof to the case where $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$. Denote by $(S_{i,j}^m)_j$ the sheets contained in $\mathfrak{g}_i^{(m)} = \{x \in \mathfrak{g}_i \mid \dim G_i.x = m\}$, $i = 1, 2$. One can decompose $\mathfrak{g}^{(m)}$ in a finite union of irreducible subsets as follows:

$$\mathfrak{g}^{(m)} = \bigcup_{\substack{p+q=m \\ j,k}} S_{1,j}^p \times S_{2,k}^q. \tag{1.9}$$

Pick $x \in \overline{S_{1,j}^p \times S_{2,k}^q}$ such that $x \notin S_{1,j}^p \times S_{2,k}^q$. Then (by symmetry) we may assume that $x = (x_1, x_2)$ with $x_i \in \overline{S_{1,j}^p}$, $i = 1, 2$, and $x_1 \notin S_{1,j}^p$. It follows that $x_1 \in \mathfrak{g}_1^{(p')}$, $x_2 \in \mathfrak{g}_1^{(q')}$ where $p' < p$ and $q' \leqslant q$, which implies $p' + q' \neq m$ and $x \notin \mathfrak{g}^{(m)}$. Therefore, (1.9) gives a decomposition of $\mathfrak{g}^{(m)}$ into irreducible closed subsets of $\mathfrak{g}^{(m)}$. We want to show that (1.9) is the decomposition of $\mathfrak{g}^{(m)}$ into irreducible components. Suppose that $S_{1,j}^p \times S_{2,k}^q \subset S_{1,j'}^{p'} \times S_{2,k'}^{q'}$. Then, $p \leqslant p'$, $q \leqslant q'$, and, since $p + q = m = p' + q'$, we get that $p = p'$ and $q = q'$. Hence, $S_{1,j}^p \subset S_{1,j'}^{p'}$ and $S_{2,k}^q \subset S_{2,k'}^{q'}$, forcing $j = j'$ and $k = k'$. This proves that $S_{1,j}^p \times S_{2,k}^q$ is an irreducible component of $\mathfrak{g}^{(m)}$. □

Recall that, since $\mathfrak{g}$ is reductive, there exists a decomposition $\mathfrak{g} = \mathfrak{z} \times \prod_i \mathfrak{g}_i$ where $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{g}_i$ is a simple Lie algebra for all i.

Corollary 1.7.2. *The G-sheets of $\mathfrak{g}$ are the sets of the form $\mathfrak{z} \times \prod_i S_i$ where each S_i is a G_i-sheet of $\mathfrak{g}_i$.*

Proof. Since $\mathfrak{z}$ is the unique sheet contained in $\mathfrak{z}$, the claim follows from Lemma 1.7.1. □

The previous corollary allows us to restrict to the case when $\mathfrak{g}$ is simple. Furthermore, it shows that the study of sheets of $\mathfrak{g}$ and of $[\mathfrak{g}, \mathfrak{g}]$ are obviously related by adding the centre. Therefore, we may for instance work with $\mathfrak{g} = \mathfrak{gl}_n$ to study of the $\mathfrak{sl}_n$-case.

2 Symmetric Lie algebras

We now turn to the symmetric case. We will denote a symmetric Lie algebra either by $(\mathfrak{g}, \theta)$, $(\mathfrak{g}, \mathfrak{k})$ or $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$, where: θ is an involution of $\mathfrak{g}$, $\mathfrak{k}$ (resp. $\mathfrak{p}$) is the $+1$(resp. -1)-eigenspace of θ in $\mathfrak{g}$. Then, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k}$ is a Lie subalgebra and $\mathfrak{p}$ is a $\mathfrak{k}$-module under the adjoint action. Recall from §1.1 that K is the connected subgroup of G such that $\mathrm{Lie}(K) = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ and that K is the connected component of

$$G^{\theta} = \{g \in G \mid g \circ \theta = \theta \circ g\} = N_G(\mathfrak{k}). \tag{2.1}$$

Sheets and Jordan classes can naturally be defined in this setting, see [TY, 39.5 & 39.6]. One has, cf. [KR, Proposition 5],

$$\dim K.x = \frac{1}{2} \dim G.x \quad \text{for all } x \in \mathfrak{p}$$

and we set:

$$\mathfrak{p}^{(m)} := \{x \in \mathfrak{p} \mid \dim K.x = m\} \subset \mathfrak{g}^{(2m)}.$$

Definition 2.0.1. The *K-sheets* of $(\mathfrak{g}, \theta)$ are the irreducible components of the $\mathfrak{p}^{(m)}$, $m \in \mathbb{N}$. Let $x = s + n$ (where $s, n \in \mathfrak{p}$) be the Jordan decomposition of an element $x \in \mathfrak{p}$. The *Jordan K-class* of x, or J_K-class of x, is the set

$$J_K(x) := K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^{\bullet} + n) \subset \mathfrak{p}.$$

It is easily seen that $\mathfrak{p}$ is the finite disjoint union of its J_K-classes and that a K-sheet is the union of the J_K-classes it contains [TY, 39.5.2].

There exists a symmetric analogue to the notion of $\mathfrak{sl}_2$-triple. An $\mathfrak{sl}_2$-triple (e, h, f) is called *normal* if $e, f \in \mathfrak{p}$ and $h \in \mathfrak{k}$. Similarily to the Lie algebra case, there is a bijection between K-orbits of nilpotent elements and K-orbits of normal $\mathfrak{sl}_2$-triples, see [KR, Proposition 4] or [TY, 38.8.5].

Any semisimple symmetric Lie algebra can be decomposed as $(\mathfrak{g}, \theta) = \prod_i (\mathfrak{g}_i, \theta_{|\mathfrak{g}_i})$ where $(\mathfrak{g}_i, \theta_{|\mathfrak{g}_i})$ is a symmetric Lie subalgebra of one of the following two types:

(a) $\mathfrak{g}_i$ simple;

(b) $\mathfrak{g}_i = \mathfrak{g}_i^1 \oplus \mathfrak{g}_i^2$, with $\mathfrak{g}_i^j$ simple, $\theta_{|\mathfrak{g}_i^j}$ isomorphism from $\mathfrak{g}_i^j$ onto $\mathfrak{g}_i^{3-j}$, $j = 1, 2$.

Each $(\mathfrak{g}_i, \theta_{|\mathfrak{g}_i})$ is called an irreducible factor of $(\mathfrak{g}, \theta)$; this decomposition is unique (up to permutation of the factors).

2.1 Type 0

When $(\mathfrak{g}, \theta)$ is the sum of two simple factors as in the above case (b), then $\mathfrak{g}$ is said to be of "type 0". We slightly enlarge this definition by saying that a pair $(\mathfrak{g}, \theta)$ is a *symmetric pair of type* 0 if

$$\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}', \quad \theta(x, y) = (y, x), \quad \mathfrak{k} = \{(x, x) \mid x \in \mathfrak{g}'\}, \quad \mathfrak{p} = \{(x, -x) \mid x \in \mathfrak{g}'\},$$

where $\mathfrak{g}'$ is only assumed to be reductive. Recall the following easy observations. Let pr_1 be the projection on the first coordinate. Via pr_1, the Lie algebra $\mathfrak{k}$ is isomorphic to $\mathfrak{g}'$, thus K is isomorphic to the adjoint group G' of $\mathfrak{g}'$. Moreover, the K-module $\mathfrak{p}$ is isomorphic to the G'-module $\mathfrak{g}'$. If Y is a subset of $\mathfrak{p}$, we set

$$\phi(Y) = \mathrm{pr}_1(Y) \times \mathrm{pr}_1(-Y) \subset \mathfrak{g}.$$

Lemma 2.1.1. (i) *The G-sheets of $\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}'$ are the $S' \times S''$ where S' and S'' are G'-sheets of $\mathfrak{g}'$.*
(ii) *The sets $\{(x, -x) \mid x \in S'\}$, where S' is a G'-sheet of $\mathfrak{g}'$, are the K-sheets of $\mathfrak{p}$.*

Proof. (i) This is a particular case of Lemma 1.7.1.
(ii) Note that an element $(x, -x)$ belongs to $\mathfrak{p}^{(m)}$ if, and only if, x is in $(\mathfrak{g}')^{(m)}$. This shows that a K-sheet of $\mathfrak{p}^{(m)}$ is contained in a set of the form $(S' \times S'') \cap \mathfrak{p}$ where S' and S'' are sheets of $(\mathfrak{g}')^{(m)}$. Observe now that Jordan classes, and consequently sheets, are stable under the transformation $x \mapsto -x$. This implies that

$$(S' \times S'') \cap \mathfrak{p} = \{(x, -x) \mid x \in S' \cap S''\}.$$

In particular, we have $(S' \times S'') \cap \mathfrak{p} \subset (S' \times S') \cap \mathfrak{p}$, where $(S' \times S') \cap \mathfrak{p}$ is an irreducible subset of $\mathfrak{p}^{(m)}$. The result then follows from: $\{(x, -x) \mid x \in S'\} \subset \overline{\{(x, -x) \mid x \in S''\}}$ if, and only if, $S' \subset \overline{S''}$. $\square$

Proposition 2.1.2. (i) *If Y is a K-orbit (resp. a J_K-class or a K-sheet) of $\mathfrak{p}$, then $\phi(Y)$ is a G-orbit (resp. a J_G-class or a G-sheet) of $\mathfrak{g}$.*
(ii) *If Z is a G-orbit (resp. a J_G-class) of $\mathfrak{g}$ intersecting $\mathfrak{p}$, then $Z \cap \mathfrak{p}$ is a K-orbit (resp. a J_K-class) of $\mathfrak{p}$*
(iii) *Distinct sheets of $\mathfrak{g}'$ have an empty intersection if, and only if, the intersection of each G-sheet of $\mathfrak{g}$ with $\mathfrak{p}$ is either empty or a single K-sheet .*

Proof. Let $x = s + n \in \mathfrak{g}'$ and set $y = (x, -x) \in \mathfrak{p}$, $Y = K.y$; then $\mathrm{pr}_1(Y) = G'.x$ and

$$G.y = (G' \times G').(x, -x) = (G'.x, G'.(-x)) = \phi(Y).$$

Conversely, if $Z = G.y$ then $Z \cap \mathfrak{p} = \{(z, -z) \mid z \in G'.x\} = K.y$.
Now, let $Y = J_K(y)$. We have $\mathrm{pr}_1(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)) = \mathfrak{c}_{\mathfrak{g}'}((\mathfrak{g}')^s)$ and the equivariance of the isomorphism $\mathrm{pr}_1 : \mathfrak{p} \to \mathfrak{g}'$ shows that the regularity condition is preserved. It follows that $\mathrm{pr}_1(Y) = G'.(\mathfrak{c}_{\mathfrak{g}'}(\mathfrak{g}'^s)^\bullet + n)$ and

$$J_G(y) = (G' \times G').(\mathfrak{c}_{\mathfrak{g}'}(\mathfrak{g}'^s)^\bullet + n, -\mathfrak{c}_{\mathfrak{g}'}(\mathfrak{g}'^s)^\bullet - n) = \phi(Y).$$

Conversely, if $Z = J_G(y)$, then $Z \cap \mathfrak{p} = \phi(Y) \cap \mathfrak{p} = \{(z, -z) \mid z \in K.(\mathfrak{c}_{\mathfrak{g}'}(\mathfrak{g}'^s)^\bullet + n)\} = J_K(y)$.
If Y is an irreducible component of $\mathfrak{p}^{(m)}$, one has $\mathrm{pr}_1(Y) = \mathrm{pr}_1(\Bbbk^* Y) = \mathrm{pr}_1(-Y)$ and $\mathrm{pr}_1(Y)$ is a G'-sheet of $\mathfrak{g}'$. As $\phi(Y)$ is an irreducible subset of $\mathfrak{g}^{(2m)}$, there exists a G-sheet S containing

$\phi(Y)$. Then, S decomposes as the product of two G'-sheet of $\mathfrak{g}'$ and therefore $S = \phi(Y)$. This ends the proof of (i) and (ii)

(iii) Let Z be a G-sheet of $\mathfrak{g}$ and write Z as the product of two G'-sheets of $\mathfrak{g}'$, say $Z = Z_1 \times Z_2$. If $(x, -x) \in Z$, it follows that $x \in Z_1 \cap Z_2$ and, in particular, $Z_1 \cap Z_2 \neq \emptyset$. If $Z_1 = Z_2$, then Lemma 2.1.1 shows that $Z \cap \mathfrak{p}$ is a K-sheet. Otherwise, one has $Z \cap \mathfrak{p} \subsetneq (Z_1 \times Z_1) \cap \mathfrak{p}$ and $Z \cap \mathfrak{p}$ is not a K-sheet of $\mathfrak{p}$. □

Since a G'-sheet of $\mathfrak{g}'$ contains exactly one nilpotent orbit of $\mathfrak{g}'$, two G'-sheets of $\mathfrak{g}'$ have a non-empty intersection if and only if they contain the same nilpotent orbit (cf. [TY, 39.3.2]). A necessary and sufficient condition for $\mathfrak{g}'$ to have intersecting sheets is therefore to have more sheets than nilpotent orbits. Using [Boh] one can show that there are only two cases where sheets are in bijection with nilpotent orbits: when $\mathfrak{g}'$ is of type A or D_4. Therefore we have:

Corollary 2.1.3. *Any G-sheet of $\mathfrak{g}$ intersects $\mathfrak{p}$ along one K-sheet if and only if the simple factors of $\mathfrak{g}'$ are of type* A *or* D_4.

The next (easy) result is true in type 0, but false in general.

Proposition 2.1.4. *Let S_G be a G-sheet of $\mathfrak{g}$ intersecting $\mathfrak{p}$. Let $\mathscr{S} = (e, h, f)$ be a normal $\mathfrak{sl}_2$-triple containing a nilpotent element $e \in S_G \cap \mathfrak{p}$. Then, if $e + X(S_G, \mathscr{S}) = (e + \mathfrak{g}^f) \cap S_G$, one has*

$$S_G \cap \mathfrak{p} = K.(e + X(S_G, \mathscr{S}) \cap \mathfrak{p}).$$

Proof. Write $S_G = S_1 \times S_2$ with S_1, S_2 sheets of $\mathfrak{g}'$(cf. Lemma 2.1.1) and set $e = (e', -e')$, $f = (f', -f')$, $e', f' \in \mathfrak{g}'$. Recall that pr_1 yields an isomorphism between $\mathfrak{p}$ and $\mathfrak{g}'$ and that $\mathrm{pr}_1(S_G \cap \mathfrak{p}) = S_1 \cap S_2$. If $X_i \subset \mathfrak{g}'$ is defined by $(e' + X_i) = (e' + \mathfrak{g}'^{f'}) \cap S_i$, one has $\mathrm{pr}_1(e + X \cap \mathfrak{p}) = e' + X_1 \cap X_2$. Moreover, $\mathrm{pr}_1(K.(e + X \cap \mathfrak{p})) = G'.(e' + X_1 \cap X_2) = S_1 \cap S_2 = \mathrm{pr}_1(S_G \cap \mathfrak{p})$. Since $\mathrm{pr}_{1|\mathfrak{p}}$ is an isomorphism, we get the desired result. □

2.2 Root systems and semisimple elements

Let $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ be a semisimple symmetric Lie algebra associated to the involution θ. Fix a Cartan subspace $\mathfrak{a}$ of $\mathfrak{p}$; recall that the *rank* of the symmetric pair $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{g}, \theta)$ is $\mathrm{rk}(\mathfrak{g}, \theta) = \dim \mathfrak{a}$. Let $\mathfrak{d}$ be a Cartan subalgebra of $\mathfrak{c}_{\mathfrak{k}}(\mathfrak{a})$. Then, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{d}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$ ([TY, 37.5.2]). If $V = \mathfrak{h}^*$ and σ denotes the transpose of θ, one can consider the σ-stable root system $R = R(\mathfrak{g}, \mathfrak{h}) \subset V$ and we set (see [TY, 36.1]):

$$V' = \{x \in \mathfrak{h}^* \mid \sigma(x) = x\} = \{x \mid x_{|\mathfrak{a}} = 0\},$$
$$V'' = \{x \in \mathfrak{h}^* \mid \sigma(x) = -x\} = \{x \mid x_{|\mathfrak{d}} = 0\},$$
$$R^0 = R \cap V' = \{\alpha \in R \mid \sigma(\alpha) = \alpha\}, \quad R^1 = \{\alpha \in R \mid \sigma(\alpha) \neq \alpha\}.$$

Recall that R^0 is a root system. One has $V = V' \oplus V''$; more precisely, $x \in V$ decomposes as $x = x' + x''$, where $x' = \frac{1}{2}(x + \sigma(x)) \in V'$, $x'' = \frac{1}{2}(x - \sigma(x)) = x_{|\mathfrak{a}} \in V''$. When $x \in R$ is a root,

x'' is called its restricted root. Set:

$$S = \{\alpha'' \mid \alpha \in R^1\}.$$

Then, $S \subset \mathfrak{a}^*$ is a (not necessarily reduced) root system, see [TY, 36.2.1], which is called the *restricted root system* of $(\mathfrak{g}, \theta)$. We denote by W, resp. W_S, the Weyl group of the root system R, resp. S, and we set

$$W_\sigma = \{w \in W \mid w \circ \sigma = \sigma \circ w\}.$$

If $B \subset R$ is a fundamental system (i.e. a basis of R), denote by R_+ (resp. R_-) the set of positive (resp. negative) roots associated to B. In order to define the Satake diagram of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ one needs to work with some special fundamental systems for R. Setting

$$R^1_\pm = R^1 \cap R_\pm$$

one can give the following definition:

Definition 2.2.1. ([TY, 36.1.4], [Ar, 2.8]) A σ-fundamental system $B \subset R$ is a fundamental system satisfying the following conditions:

(i) $\sigma(R^1_+) = R^1_-$;

(ii) If $\alpha \in R^1_+$, $\beta \in R$ and $\alpha - \beta \in V'$, then $\beta \in R^1_+$;

(iii) $(R^1_+ + R^1_+) \cap R \subset R^1_+$;

Let $V_\mathbb{Q}$ be the $\mathbb{Q}$-vector space spanned by R; then $V_\mathbb{Q} = V'_\mathbb{Q} \oplus V''_\mathbb{Q}$ where $V'_\mathbb{Q} = V_\mathbb{Q} \cap V'$, resp. $V''_\mathbb{Q} = V_\mathbb{Q} \cap V''$, are $\mathbb{Q}$-forms of V', resp. V'' (cf. [TY, proof of 36.1.4]). Denote by $\mathfrak{a}_\mathbb{Q}$ the $\mathbb{Q}$-form of $\mathfrak{a}$ given by the dual of $V''_\mathbb{Q}$. The choice of a $\mathbb{Q}$-basis $C = (e_1, \dots, e_l)$ of $V_\mathbb{Q}$ gives rise to a lexicographic ordering $\prec$ on $V_\mathbb{Q}$ and, therefore, to a set of positive roots $R_{+,C} = \{\alpha \in R \mid \alpha \succ 0\}$. Recall [TY, 18.7] that for each choice of such a basis C, there exists a unique fundamental system B_C such that $R_{+,C}$ is the set of positive roots with respect to B. The existence of a σ-fundamental system is ensured by the next lemma, which provides all the σ-fundamental systems, see Proposition 2.2.3(iv).

Lemma 2.2.2. *Let $(e_1, \dots, e_p)$, resp. $(e_{p+1}, \dots, e_l)$, be a basis of $V''_\mathbb{Q}$, resp. $V'_\mathbb{Q}$, and set $C = (e_1, \dots, e_l)$. Then B_C is a σ-fundamental system such that $B^0_C = B_C \cap V'$ is a fundamental system of R^0.*

Proof. By [TY, 36.1.4] B_C is a σ-fundamental system. The second statement follows from the fact that $B_C \cap V'$ is the set of simple roots associated to the lexicographic ordering associated to the basis $(e_{p+1}, \dots, e_l)$. □

Proposition 2.2.3. (i) *The map $w \mapsto w_{|V''}$ induces a surjective homomorphism $W_\sigma \to W_S$ whose kernel is W^0, the Weyl group of R^0.*

(ii) *For $x \in V''_{\mathbb{Q}}$, one has $W_S.x = W.x \cap V''_{\mathbb{Q}}$. Dually, $W_S.a = W.a \cap \mathfrak{a}_{\mathbb{Q}}$ for all $a \in \mathfrak{a}_{\mathbb{Q}}$.*
(iii) *Let B be a σ-fundamental system. Then, the restricted fundamental system $B'' = \{\alpha'' \mid \alpha \in B\}$ is a fundamental system of the restricted root system S.*
(iv) *W_σ acts transitively on the set of σ-fundamental systems.*

Proof. Claims (i) and (ii) are proved in [TY, 36.2.5, 36.2.6], while (iii) and (iv) can be found in [Ar, 2.8 and 2.9]. □

Remarks 2.2.4. (1) The restriction to $\mathfrak{a}$ yields an isomorphism $N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \xrightarrow{\sim} W_S$, cf. [TY, 38.7.2].
(2) Let $w \in W_\sigma$, then there exists $k \in K$ such that $k_{|\mathfrak{h}} = w$. This can be shown as follows. Recall that $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{d}$, where $\mathfrak{d}$ is a Cartan subalgebra of $\mathfrak{u} = \mathfrak{c}_{\mathfrak{k}}(\mathfrak{a})$. Note that $w.\mathfrak{a} = \mathfrak{a}$ and $w.\mathfrak{d} = \mathfrak{d}$. Pick $k_1 \in K$ such that $k_{1|\mathfrak{a}} = w_{|\mathfrak{a}} \in W_S$. Let $U \subset C_K(\mathfrak{a})$ be the connected subgroup of K with Lie algebra $\mathfrak{u}$. The Weyl group of the root system $R^0 = R(\mathfrak{u}, \mathfrak{d})$ is $W^0 \cong N_U(\mathfrak{d})/Z_U(\mathfrak{d})$, see [TY, 38.2.1]. By composing k_1 with an element of U we may assume that $k_1.\mathfrak{h} = \mathfrak{h}$ and $k_{1|\mathfrak{a}} = w_{|\mathfrak{a}}$. Set $w_0 = (w \circ k_1^{-1})_{|\mathfrak{h}} \in W$; one has $w_{0|\mathfrak{a}} = \mathrm{Id}_{\mathfrak{a}}$, therefore $w_0 \in W^0$ and we can find $k_0 \in N_U(\mathfrak{d})$ such that $k_{0|\mathfrak{d}} = w_{0|\mathfrak{d}} = w_{|\mathfrak{d}} \circ k_1^{-1}{}_{|\mathfrak{d}}$. Setting $k = k_0 k_1 \in K$ we obtain $k_{|\mathfrak{a}} = k_{1|\mathfrak{a}} = w_{|\mathfrak{a}}$ and $k_{|\mathfrak{d}} = k_{0|\mathfrak{d}} \circ k_{1|\mathfrak{d}} = w_{|\mathfrak{d}}$, thus $k_{|\mathfrak{h}} = w$.

Fix a σ-fundamental system B; from the Dynkin diagram D associated to B one can construct the Satake diagram $\bar{D}$ of $(\mathfrak{g}, \theta)$ as follows. The nodes α of D such that $\alpha'' = 0$ are colored in black, the other nodes being white; two white nodes $\alpha \neq \beta$ of D such that $\alpha'' = \beta''$ are related by a two-sided arrow. This defines the new diagram $\bar{D}$. Recall that the Satake diagram of $(\mathfrak{g}, \theta)$ does not depend on the choice of the σ-fundamental system B, and that two semisimple symmetric Lie algebras are isomorphic if and only if they have the same Satake diagram (cf. [Ar, Theorem 2.14]). A classification of symmetric Lie algebras together with their Satake diagrams and restricted root systems is given in [He1, Ch. X].

We now recall the (well-known) links between G-conjugacy and W-conjugacy, and their analogues for a symmetric Lie algebra.

Lemma 2.2.5. (i) *Two elements of $\mathfrak{h}$ (resp. $\mathfrak{a}$) are G(resp. K)-conjugate if and only if they are W(resp. W_S or, equivalently, W_σ)-conjugate.*
(ii) *Let $x, y \in \mathfrak{h}$ (resp. $x, y \in \mathfrak{a}$), then the Levi factors $\mathfrak{g}^x$ and $\mathfrak{g}^y$ are G(resp. K)-conjugate if, and only if, they are W(resp. W_S or, equivalently, W_σ)-conjugate.*

Proof. (i) We write the proof for $x, y \in \mathfrak{a}$. Thanks to [TY, 29.2.3 & 37.4.10] applied to $(\mathfrak{g}^y, \mathfrak{k}^y)$, the elements x, y are K-conjugate if, and only if, there exists an element $g \in K$ such that $g.x = y$, $g.\mathfrak{h} = \mathfrak{h}$. Then g induces an element of W, and therefore of W_σ since $g \circ \sigma = \sigma \circ g$. Observe finally that Proposition 2.2.3(i) implies the equivalence of W_σ and W_S-conjugacy. Conversely, [TY, 38.7.2] shows that the conjugation under W_S implies the K-conjugation.
(ii) The proof is analogue to (i). Indeed, one can show (using the conjugacy of Cartan subalgebras) that $G.\mathfrak{g}^x = G.\mathfrak{g}^y$ (resp. $K.\mathfrak{g}^x = K.\mathfrak{g}^y$) if, and only if, there exists $g \in G$ (resp. K) such that $g.\mathfrak{g}^x = \mathfrak{g}^y$ and $g.\mathfrak{h} = \mathfrak{h}$. □

In general, if $x \in \mathfrak{p}$, the intersection of $G.x$ with $\mathfrak{p}$ contains more than one orbit (cf. [TY, 38.6.1(i)]). But, when x is semisimple one can prove the following result, for which we provide a proof since we did not find a reference in the literature.

Proposition 2.2.6. *Let $s \in \mathfrak{p}$ be semisimple. Then, $G.s \cap \mathfrak{p} = K.s$.*

Proof. Recall that any semisimple element of $\mathfrak{p}$ is K-conjugate to an element of $\mathfrak{a}$, cf. [TY, 37.4.10]. Therefore, by Lemma 2.2.5(i), it suffices to show that the property (ii) of Proposition 2.2.3 holds for all $a \in \mathfrak{a}$, i.e. $W_S.a = W.a \cap \mathfrak{a}$. Denote by $\mathbb{L}$ one of the fields $\mathbb{Q}$ or $\Bbbk$. For $(w, w') \in W \times W_S$, define linear subspaces of $\mathfrak{a}_{\mathbb{L}} = \mathfrak{a}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{L}$ by:

$$E_{\mathbb{L}}^{w,w'} = \ker_{\mathfrak{a}_{\mathbb{L}}}(w - w') = \{a \in \mathfrak{a}_{\mathbb{L}} \mid w.a = w'.a\}, \quad E_{\mathbb{L}}^{w} = w^{-1}(\mathfrak{a}_{\mathbb{L}}) \cap \mathfrak{a}_{\mathbb{L}}.$$

From Proposition 2.2.3(ii) one gets that $E_{\mathbb{Q}}^{w} = \bigcup_{w' \in W_S} E_{\mathbb{Q}}^{w,w'}$; thus, there exists $w' \in W_S$ such that $E_{\mathbb{Q}}^{w} = E_{\mathbb{Q}}^{w,w'}$. The flatness of $- \otimes_{\mathbb{Q}} \Bbbk$ yields:

$$E_{\Bbbk}^{w,w'} = E_{\mathbb{Q}}^{w,w'} \otimes_{\mathbb{Q}} \Bbbk, \quad E_{\Bbbk}^{w} = E_{\mathbb{Q}}^{w} \otimes_{\mathbb{Q}} \Bbbk.$$

Therefore, for any $w \in W$, there exists $w' \in W_S$ such that $w'_{|E_{\Bbbk}^{w}} = w_{|E_{\Bbbk}^{w}}$. It follows that Proposition 2.2.3(ii) is satisfied for all $a \in \mathfrak{a} = \mathfrak{a}_{\Bbbk}$. □

Consequence. Proposition 2.2.6 yields a bijection between K-orbits of semisimple elements of $\mathfrak{p}$ and G-orbits of semisimple elements intersecting $\mathfrak{p}$.

Recall [Ko, KR] that the set of semisimple G(resp. K)-orbits is parameterized by the categorical quotient $\mathfrak{g}/\!\!/G$ (resp. $\mathfrak{p}/\!\!/K$), and that $\Bbbk[\mathfrak{g}/\!\!/G] \cong \Bbbk[\mathfrak{h}/W] = S(\mathfrak{h}^*)^W$, $\Bbbk[\mathfrak{p}/\!\!/K] \cong \Bbbk[\mathfrak{a}/W_S] = S(\mathfrak{a}^*)^{W_S}$. The previous consequence can then be interpreted as follows.
Let γ be the map which associates to the W_S-orbit of $a \in \mathfrak{a}$, the orbit $W.a \subset \mathfrak{h}$; hence, $\gamma : \mathfrak{a}/W_S \to \mathfrak{h}/W$. Define $Z = \gamma(\mathfrak{a}/W_S) \subset \mathfrak{h}/W$ to be the image of γ and let $\phi : \mathfrak{a}/W_S \to Z$ be the induced surjective map. Write $\gamma = \iota \circ \phi$, where $\iota : Z \to \mathfrak{h}/W$ is the natural inclusion. Since γ is a finite morphism, Z is closed and ϕ is a finite surjective morphism. We then have the two following commutative (dual) diagrams:

$$\begin{tikzcd} \mathfrak{a}/W_S \arrow[r, "\phi"] \arrow[dr, "\gamma"'] & Z \arrow[d, "\iota"] \\ & \mathfrak{h}/W \end{tikzcd} \qquad \begin{tikzcd} S(\mathfrak{a}^*)^{W_S} & \Bbbk[Z] \arrow[l, "\phi^*"'] \\ & S(\mathfrak{h}^*)^W \arrow[ul, "\gamma^*"] \arrow[u, "\iota^*"'] \end{tikzcd}$$

The significance of Proposition 2.2.6 is that γ is injective; equivalently, ϕ is bijective.
Since $\mathfrak{a}/W_S$ is a normal variety ($S(\mathfrak{a}^*)^{W_S}$ is a polynomial ring), and $S(\mathfrak{a}^*)^{W_S}$ is an integral extension of $\Bbbk[Z]$ (ϕ is finite), [TY, 17.4.4] yields the following:

Corollary 2.2.7. *The morphism $\phi : \mathfrak{a}/W_S \to Z$ is a bijective birational map, and $\mathfrak{a}/W_S$ is the normalization of Z.*

One must observe that the injective map ϕ^* is not surjective, *i.e.* Z is not normal, in general. This question has been studied in [He2, He3, Ri2, Pa4]. The notation being as in [He1, Ch. X], the results obtained in the previous references show that ϕ is an isomorphism when $\mathfrak{g}$ is of classical type, and in the exceptional cases of type EI, EII, EV, EVI, EVIII, FI, FII, G. In cases EIII, EIV, EVII, EIX, it is known that ϕ^* (or, equivalently, γ^*) is not surjective, cf. [He2, Ri2].

Remark 2.2.8. By standard arguments one can see that the results obtained in 2.2.4, 2.2.5 and 2.2.6 remain true when $(\mathfrak{g}, \theta)$ is a *reductive* symmetric Lie algebra.

2.3 Property (L)

Let $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$, $\mathfrak{a}, \mathfrak{h}$, R, R^0, R^1, S be as in 2.2, and fix a σ-fundamental system B of R (cf. Definition 2.2.1). The next definition introduces an important property in order to study the K-conjugacy classes of Levi factors of the form $\mathfrak{g}^s$, $s \in \mathfrak{p}$ semisimple. Observe that $(\mathfrak{g}^s, \mathfrak{k}^s)$ is a symmetric Lie algebra, that we will call a *subsymmetric pair.*

Definition 2.3.1. The pair $(\mathfrak{g}, \mathfrak{k})$ satisfies the property (L) if, for all semisimple elements $s, u \in \mathfrak{p}$:

$$\{\exists\, g \in G,\ g.\mathfrak{g}^s = \mathfrak{g}^u\} \iff \{\exists\, k \in K,\ k.\mathfrak{g}^s = \mathfrak{g}^u\}. \tag{L}$$

Remark 2.3.2. More generally, when $(\mathfrak{g}, \theta)$ is a reductive symmetric Lie algebra, the condition (L) holds if and only if it holds for $([\mathfrak{g}, \mathfrak{g}], \theta)$.

The aim of this section is to prove that the property (L) holds for any reductive symmetric Lie algebra (cf. Theorem 2.3.7). We are going to show that it is sufficient to check (L) for some Levi factors $\mathfrak{g}^s$ of a particular type, cf. Corollary 2.3.6.

Definition 2.3.3. Let $s \in \mathfrak{h}_{\mathbb{Q}}$ be in the positive Weyl chamber defined by B. One says the standard Levi factor $\mathfrak{g}^s$ *arises from* $\mathfrak{p}$ if one can choose s in $\mathfrak{a}_{\mathbb{Q}}$.

Recall from Section 1 that there is a natural one to one correspondence between standard Levi factors and subsets of B. In this correspondence, to a Levi factor $\mathfrak{l}$ one associates the subset

$$I_{\mathfrak{l}} := \{\alpha \in B \mid \alpha(s) = 0\}$$

where s is any element in $(\mathfrak{g}^{\mathfrak{l}})^{\bullet}$. Conversely, from any subset $I \subset B$ one gets a Levi subalgebra by setting:

$$\mathfrak{l}_I = \mathfrak{h} \oplus \left(\oplus_{\alpha \in \langle I \rangle} \mathfrak{g}^{\alpha}\right)$$

where $\langle I \rangle = \mathbb{Z}I \cap R$. Remark that $\mathfrak{g}^{\mathfrak{l}_I} = \{h \in \mathfrak{h} : \alpha(h) = 0 \text{ for all } \alpha \in I\}$.

Let D be the Dynkin diagram defined by B and denote by $\bar{D}$ the associated Satake diagram. Let $B^0 \subset B$ be the set of black nodes of $\bar{D}$; recall that B^0 is a fundamental system of R^0 (cf. Lemmas 2.2.2 and 2.2.3). Set

$$B^2 = \{(\alpha_1, \alpha_2) \in B \times B : \alpha_1 \neq \alpha_2,\ \alpha_1'' = \alpha_2''\}, \quad B^3 = \left\{\alpha_1 - \alpha_2 \mid (\alpha_1, \alpha_2) \in B^2\right\} \subset \mathfrak{h}_{\mathbb{Q}}^*.$$

Thus, B^2 is the set of pairs of white nodes ($\alpha_1 \neq \alpha_2$) of $\bar{D}$ connected by a two-sided arrow (note that $(\alpha_1, \alpha_2) \in B^2 \iff (\alpha_2, \alpha_1) \in B^2$). Denote by $\overline{B^2} \subset B$ the set of all nodes pointed by such an arrow, i.e. $\overline{B^2} = \{\alpha \in B : \exists \beta \in B,\ (\alpha, \beta) \in B^2\}$. A subset $I \subset B$ is said to be *stable under arrows* if $(\alpha_1, \alpha_2) \in B^2$ with $\alpha_1 \in I$ implies $\alpha_2 \in I$.

Remark 2.3.4. The subspace $\mathfrak{a}_{\mathbb{Q}} \subset \mathfrak{h}_{\mathbb{Q}}$ is the intersection of the kernels of elements of $B^0 \cup B^3$. A standard Levi factor $\mathfrak{l}$ arises from $\mathfrak{p}$ if, and only if, $I_{\mathfrak{l}}$ is stable under arrows and contains B^0.

We now want to describe the subalgebra $\mathfrak{g}^s$ when $s \in \mathfrak{a}$ semisimple. Set

$$E_s = \{\varphi \in \mathfrak{h}_{\mathbb{Q}}^* = V_{\mathbb{Q}} : \varphi(s) = 0\}, \quad R_s = E_s \cap R.$$

Then, R_s is a root subsystem of R (cf. [TY, 18.2.5]) and, with obvious notation, the $\mathbb{Q}$-vector space F_s spanned by R_s decomposes as $F_s' \oplus F_s''$. The restriction to $\mathfrak{h}_{s,\mathbb{Q}} = \mathfrak{h}_{\mathbb{Q}} \cap [\mathfrak{g}^s, \mathfrak{g}^s]$ identifies F_s with $\mathfrak{h}_{s,\mathbb{Q}}^*$ and R_s with the root system of $(\mathfrak{g}^s, \mathfrak{k}^s)$. One can therefore apply to R_s the results of section 2.2.
Let S_s be the restricted root system of R_s. As $s \in \mathfrak{a}$, one has:

$$S_s = \{x'' \mid x \in R^1, x(s) = 0\} = \{x'' \mid x \in R^1, x''(s) = 0\} = S \cap F_s''. \tag{2.2}$$

Let B_s be a σ-fundamental system of R_s. One can write $B_s = B_s^0 \sqcup B_s^1$ with $B_s^0 \subset R^0$, $B_s^1 \subset R^1$ and we denote by B_s'' the restricted fundamental system of S_s associated to B_s.
We can now prove the following result:

Proposition 2.3.5. *Each Levi factor $\mathfrak{g}^s$, $s \in \mathfrak{p}$, is K-conjugate to a standard Levi factor that arises from $\mathfrak{p}$.*

Proof. Since the element $s \in \mathfrak{p}$ is semisimple, it is K-conjugate to an element of $\mathfrak{a}$ and we may as well suppose that $s \in \mathfrak{a}$. We will use the previous notation relative to R_s, S_s and a fixed σ-fundamental system $B_s \subset R_s$.

We first show that there exists $w \in W_\sigma$ such that $B_s \subset w.B$. Since $V_{\mathbb{Q}}' \subset E_s$ one has $R^0 \subset R_s$, and B_s^0 being a fundamental system of the root system R^0, it can be conjugated to B^0 by an element of W^0. As B_s'' is a fundamental system of $S_s = S \cap F_s''$ (see (2.2)), [TY, 18.7.9(ii)] implies that B_s'' is a W_S-conjugate of a subset of B''. Combining these two facts and Lemma 2.2.3(i), one gets the existence of $w \in W_\sigma$ such that $B_s^0 = w.B^0$ and $B_s'' \subset w.B''$.
We claim that $B_s \subset w.B$, i.e. $B_s^1 \subset w.B$. Let $\alpha \in B_s^1$. Since $w.B$ is a σ-fundamental system of R, there exist integers $(n_\gamma)_{\gamma \in w.B}$, of the same sign, such that $\alpha = \sum_{\gamma \in w.B} n_\gamma \gamma$ and $\alpha'' = \sum_{\gamma \in w.B^1} n_\gamma \gamma''$. As $\alpha'' \in w.B''$, the n_γ's must be positive and there exists a unique $\beta \in w.B^1$ such that: $\alpha'' = \beta''$, $n_\beta = 1$, $n_\gamma = 0$ for $\gamma \in w.B^1 \smallsetminus \{\beta\}$. One then gets $\beta = \alpha - \sum_{\gamma \in w.B^0 = B_s^0} n_\gamma \gamma$, hence $\beta \in R_s$. But B_s is a fundamental system of R_s, thus the previous decomposition of β as a sum of positive and negative elements of B_s forces $n_\gamma = 0$ for $\gamma \in B_s^0$. Therefore $\alpha = \beta \in w.B$, as desired.

Pick $\dot{w} \in K$ such that $\dot{w}.s = w.s$, see Remark 2.2.4(2); replacing $\mathfrak{g}^s$ by $\mathfrak{g}^{\dot{w}.s}$ we may assume that $w = \mathrm{Id}$ and $B_s \subset B$. Define $t \in \mathfrak{h}_{\mathbb{Q}}$ by the conditions: $\alpha(t) = 0$ for $\alpha \in B_s$ and $\beta(t) = 1$ for $\beta \in B \smallsetminus B_s$. Then, $t \in \bigcap_{\varphi \in B^0 \cup B^3} \ker \varphi = \mathfrak{a}_{\mathbb{Q}}$ (cf. Remark 2.3.4). Finally, since B_s is a fundamental system of R_s, it is easily seen that $\mathfrak{g}^t = \mathfrak{g}^s$. □

From the previous proposition one deduces the announced result:

Corollary 2.3.6. *The property* (L) *is equivalent to: "Two standard Levi factors arising from* $\mathfrak{p}$ *are* G*-conjugate if, and only if, they are* K*-conjugate".*

We can now show that (L) is satisfied by any symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k})$.

Theorem 2.3.7. (i) *If* $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$ *are two standard* G*-conjugate Levi factors arising from* $\mathfrak{p}$ *such that* $B^0 \cup \overline{B^2} \subset I_{\mathfrak{l}_1}$*, then* $\mathfrak{g}^{\mathfrak{l}_1} \subset \mathfrak{a} \subset \mathfrak{p}$ *and* $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$ *are* K*-conjugate.*
(ii) *Assume that there is no arrow in the Satake diagram of* $(\mathfrak{g}, \mathfrak{k})$*. Then, for any semisimple element* $s \in \mathfrak{p}$ *one has* $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s) = \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) \subset \mathfrak{p}$*, and the property* (L) *is satisfied by* $(\mathfrak{g}, \mathfrak{k})$*.*
(iii) *If* $(\mathfrak{g}, \mathfrak{k})$ *is irreducible of type* AIII, DI, DIII, EII, EIII*, then it satisfies* (L)*.*
(iv) *Every reductive symmetric Lie algebra satisfies the property* (L)*.*

Proof. (i) The first assertion follows from the characterization of $\mathfrak{a}$ given in Remark 2.3.4. Let $s \in (\mathfrak{g}^{\mathfrak{l}_2})^\bullet$, hence $\mathfrak{g}^s = \mathfrak{l}_2$; by hypothesis, there exists $g \in G$ such that $g.s \in (\mathfrak{g}^{\mathfrak{l}_1})^\bullet \subset \mathfrak{p}$. Proposition 2.2.6 then implies the existence of $k \in K$ such that $g.s = k.s$, thus: $\mathfrak{l}_1 = \mathfrak{g}^{k.s} = k.\mathfrak{l}_2$.
(ii) Observe that, here, $\overline{B^2} = \emptyset$. By Proposition 2.3.5 we may assume that $\mathfrak{g}^s = \mathfrak{l}$ with $s \in \mathfrak{a}_{\mathbb{Q}}$. Then, obviously, $B^0 \subset I_{\mathfrak{l}}$ and from (i) one deduces $\mathfrak{g}^{\mathfrak{l}} \subset \mathfrak{p}$. Therefore, $\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) = \mathfrak{g}^{\mathfrak{l}} \cap \mathfrak{p} = \mathfrak{g}^{\mathfrak{l}} \subset \mathfrak{p}$ (cf. [TY, 38.8.3]). The proof of (L) follows as in (i).
(iii) Let $\mathfrak{g}^{s_i}$, $s_i \in \mathfrak{a}_{\mathbb{Q}}$, $i = 1, 2$, be two standard Levi factors arising from $\mathfrak{p}$. Observe first that Proposition 2.2.5(ii) yields: $\mathfrak{g}^{s_1}, \mathfrak{g}^{s_2}$ are G-conjugate $\iff$ $\mathfrak{g}^{s_1}, \mathfrak{g}^{s_2}$ are W-conjugate, and $\mathfrak{g}^{s_1}, \mathfrak{g}^{s_2}$ are K-conjugate $\iff$ $\mathfrak{g}^{s_1}, \mathfrak{g}^{s_2}$ are W_σ-conjugate. Let B be a σ-fundamental system; denote by Φ the set of all subsets of B which contain all black nodes and which are sable under arrows. Observe that $E \in \Phi$ is equivalent to $E = I_{\mathfrak{l}}$ for some standard Levi factor $\mathfrak{l}$ arising from $\mathfrak{p}$. Therefore, by the previous remark, we need to show that two elements of Φ are W-conjugate if and only if they are W_σ-conjugate.
For $E \in \Phi$ we define a subset $\phi(E)$ of B'', the fundamental system of the restricted root system S, by setting $\phi(E) = \{\alpha'' : \alpha \in E\} \smallsetminus \{0\}$. It is easy to see that ϕ defines a bijection from Φ onto Φ'', the set of all subsets of B'', and that two elements of Φ are W_σ-conjugate if and only if their images by ϕ are W_S-conjugate. By abuse of notation, we denote by Φ/W and Φ/W_σ resp. Φ''/W_S, the set of orbits under W and W_σ, resp. W_S, of elements of Φ, resp. Φ''. Since $W_\sigma \subset W$, there exists a natural surjection π from Φ/W_σ onto Φ/W, hence $\#\Phi/W \leq \#\Phi''/W_S = \#\Phi/W_\sigma$, and we need to show that π is bijective. We have remarked above that ϕ^{-1} yields a bijection between Φ''/W_S and Φ/W_σ. Let $\delta : \Phi''/W_S \to \Phi/W$ be the surjection induced by $\pi \circ \phi^{-1}$. It remains to show that δ is injective, or, equivalently, that $\#\Phi/W \geq \#\Phi''/W_S$.

The description of ϕ, Φ and Φ'' can be deduced from [He1, p. 532]. In [BC, p. 5] are given the W-conjugacy classes of subsets of B (cf. [Dy, Theorem 5.4] for the original classification). Using these results we are now going to make a case by case comparison of Φ/W and Φ''/W_S. We mostly adopt the notation of [BC]; in particular, an element of Φ/W will be identified with a Dynkin subdiagram of the Dynkin diagram defined by B. We make a distinction between the diagrams of type $\mathrm{A}_1, \mathrm{B}_1$, resp. $2\mathrm{A}_1, \mathrm{D}_2$, resp. $\mathrm{A}_3, \mathrm{B}_3$, as in [Os, §10]. A similar notation is used for the elements of Φ''/W_S.

Cases EIII & EII: In case EIII, one finds that $\Phi/W = \{\mathrm{E}_6, \mathrm{A}_5, \mathrm{D}_4, \mathrm{A}_3\}$ and $\Phi''/W_S = \{\mathrm{B}_2, \mathrm{B}_1, \mathrm{A}_1, \emptyset\}$. In case EII, one easily sees that $\#\Phi/W = \#\Phi''/W_S = 12$.

Cases DI & DIII: By (i), one can reduce the comparison to the elements of Φ which do not contain elements of $\overline{B^2}$. Denote by $\Phi_1 \subset \Phi$ these subsets and set $\Phi_1'' = \phi(\Phi_1) \subset \Phi''$. The conjugacy classes of Φ_1, Φ_1'' are, respectively, denoted by Φ_1/W and Φ_1''/W_S.
In type DI we are then reduced to consider the case $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}_{2n}, \mathfrak{so}_{n-1} \times \mathfrak{so}_{n+1})$, where R is of type D_n and S of type B_{n-1}. Under the above notation, any element E in Φ_1/W has type $\sum_k \mathrm{A}_{i_k} + \mathrm{D}_j$ where: $i_k \in \mathbb{N}$, $\sum_k (i_k + 1) + j = n$ (the Bala-Carter conditions) and $j = 0$, $i_{k_0} = 0$ for at least one k_0 (because $E \in \Phi_1/W$). Since $i_{k_0} = 0$ is even, it follows from [BC] that there exists a unique conjugacy class in Φ_1 satisfying these properties. Then, $\delta^{-1}(E)$ contains an unique element, of type $\sum_{k \neq k_0} \mathrm{A}_{i_k} + \mathrm{B}_j$, determined by the previous conditions.
The remaining case, $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{so}_{4n+2}, \mathfrak{gl}_{2n+1})$, in type DIII is similar.

Case AIII: Here, $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}_{p+q}, \mathfrak{sl}_p \times \mathfrak{sl}_q \times \Bbbk)$ with $p \geqslant q$. The restricted root system is of type B_q (when $p > q$) or C_q (when $p = q$).
The W-conjugacy class N of an element of Φ is given by a diagram of type $\sum_i n_i \mathrm{A}_i$, where $\sum_i n_i(i+1) = p + q$ and at most one n_{i_0} is odd. One has $p - q = 1$ if i_0 does not exist, and $i_0 \geq p - q - 1$ otherwise. The W_S-conjugacy class of $\phi(N)$ is given by a diagram $\sum_i m_i \mathrm{A}_i + \mathrm{BC}_j$ defined by the following rule: $m_i = n_i/2$ if $i \neq i_0$, $m_{i_0} = (n_{i_0} - 1)/2$, and

$$\mathrm{BC}_j = \begin{cases} \mathrm{B}_{(i_0-(p-q-1))/2} & \text{if } i_0 \text{ exists and } p \neq q; \\ \mathrm{B}_0 & \text{if } p - q = 1 \text{ and } i_0 \text{ does not exists;} \\ \mathrm{C}_{(i_0+1)/2} & \text{if } p = q \text{ and } i_0 \neq 0; \\ \mathrm{C}_0 & \text{if } p = q \text{ and } i_0 = 0. \end{cases}$$

It follows from [BC] that this class depends only on the class N. Therefore δ is injective, proving (iii) in type AIII.
(iv) When $(\mathfrak{g}, \theta)$ is of type 0 there is an obvious bijection between W-conjugacy classes of elements Φ and W_S-conjugacy classes in Φ''. Since $\mathfrak{g}$ is a direct product of irreducible symmetric Lie algebras, the result then follows from (ii) and (iii). □

2.4 Jordan K-classes

Let $(\mathfrak{g}, \mathfrak{k})$ be a reductive symmetric Lie algebra. We adopt the notation of §1.3 and Definition 2.0.1. Observe the following easy result:

Lemma 2.4.1. *The intersection of a J_G-class with $\mathfrak{p}$ is either empty or the union of J_K-classes it contains.*

Proof. Let J be a Jordan G-class intersecting $\mathfrak{p}$ and $x = s + n \in J \cap \mathfrak{p}$. Then $J_K(x) = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n) \subset G.(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)^\bullet + n) = J_G(x)$. □

In Lemma 2.4.2 we fix a J_G-class J such that $J \cap \mathfrak{p} \neq \emptyset$, and an element $x = s + n \in J \cap \mathfrak{p}$. Let $\mathfrak{l} = \mathfrak{g}^s$ and $L = G^s \subset G$ be the associated Levi factors. Observe that:

$$L = Z_G(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)^\bullet). \tag{2.3}$$

Then, $(\mathfrak{g}^s, \mathfrak{k}^s)$ is a symmetric pair and $K_L = (K \cap L)^\circ \subset K^s$ acts naturally on $\mathfrak{p}^s$. Denote by $\mathcal{O}_1$ the orbit $L.n \in \mathfrak{l}$, so that $(\mathfrak{l}, \mathcal{O}_1)$ is a datum of J. Let $\mathcal{O}_i \subset \mathfrak{l}$ $(i > 1)$ be the L-orbits (if they exist) different from $\mathcal{O}_1$ such that $(\mathfrak{l}, \mathcal{O}_i)$ is a datum of J. Define nilpotent K_L-orbits in $\mathfrak{p}^s$ by

$$\mathcal{O}_i \cap \mathfrak{p}^s = \bigcup_j \mathcal{O}_i^j, \quad \mathcal{O}_i^j = K_L.n_i^j.$$

Lemma 2.4.2. (i) *One has $J \cap \mathfrak{p} = \bigcup_{i,j} K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n_i^j)$.*
(ii) *Any J_K-class contained in $J \cap \mathfrak{p}$ has dimension* $\dim \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) + \dim K.x$.

Proof. Let $y = s' + n' \in J \cap \mathfrak{p}$. Since x and y belong to the same J_G-class, $\mathfrak{g}^{s'}$ is G-conjugate to $\mathfrak{g}^s$ [TY, 39.1.3]. By Property (L), see Theorem 2.3.7, the subalgebra $\mathfrak{g}^{s'}$ is then K-conjugate to $\mathfrak{g}^s$. We can therefore assume that $s' \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet$. It follows that n' belongs to one of the orbits $K_L.n_i^j$, hence $J_K(y) = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n_i^j) \subset J \cap \mathfrak{p}$.
By [TY, 39.5.8] one knows that $\dim J_K(y) = \dim K.y + \dim \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) = \dim K.x + \dim \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) = \dim J_K(x)$. This proves (i) and (ii). □

Note that the union in Lemma 2.4.2(i) is not necessarily a disjoint union.

Lemma 2.4.3. (i) *Let $g \in G$ and $s \in \mathfrak{p}$ semisimple be such that $g.s \in \mathfrak{p}$; then $g.\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) \subset \mathfrak{p}$.*
(ii) *For $x, y \in \mathfrak{p}$ such that $G.x = G.y$, one has $G.J_K(x) = G.J_K(y)$.*

Proof. (i) By Lemma 2.2.6 there exists $k \in K$ such that $k.(g.s) = s$, hence $kg \in L = Z_G(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)^\bullet)$ (see (2.3)) and $kg.\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) = \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)$. This gives $g.\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) = k^{-1}.\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s) \subset \mathfrak{p}$.
(ii) By Lemma 2.2.6, again, we may assume that $x = s + n$ and $y = s + n'$. Then, $J_K(x) = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n)$ and $J_K(y) = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n')$. Write $y = g.x$, $g \in G$; from (2.3) it follows that $g.(s' + n) = s' + n'$ for all $s' \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet$. □

We can now describe the intersection of a J_G-class with $\mathfrak{p}$.

Theorem 2.4.4. *Let J be a Jordan G-class. The variety $J \cap \mathfrak{p}$ is smooth. The J_K-classes contained in $J \cap \mathfrak{p}$ are its (pairwise disjoint and smooth) irreducible components.*

Proof. We may obviously assume that $J\cap\mathfrak{p}\neq\emptyset$; pick $x\in J\cap\mathfrak{p}$. Recall [Bro] that J is smooth and that the tangent space T_xJ is equal to $[x,\mathfrak{g}]\oplus\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)$, see [TY, 39.2.8, 39.2.9]. By [TY, 39.5.5] there exists a dominant morphism $\mu: K\times\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^x)^{\bullet}\to J_K(x)$, $(k,u)\mapsto k.u$. Therefore $d_{(\mathrm{Id},x)}\mu(\mathfrak{k}\times\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^x))=[x,\mathfrak{k}]\oplus\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)$ (cf. [TY, 39.5.7]) is a subspace of the tangent space $T_xJ_K(x)$, and we then obtain:

$$T_x(J\cap\mathfrak{p})\subset T_xJ\cap\mathfrak{p}=([x,\mathfrak{g}]\oplus\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s))\cap\mathfrak{p}=[x,\mathfrak{k}]\oplus\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)\subset T_xJ_K(x)\subset T_x(J\cap\mathfrak{p}).$$

Thus $T_x(J\cap\mathfrak{p})=T_xJ_K(x)$ has dimension $\dim J_K(x)=\dim\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)+\dim K.x$. By Lemma 2.4.2(ii), this dimension does not depend on the element x chosen in $J\cap\mathfrak{p}$. Therefore $J_K(x)$, $J\cap\mathfrak{p}$ are smooth and each element of $J\cap\mathfrak{p}$ belongs to a unique irreducible component (see, for example, [TY, 17.1.3]). Then, Lemma 2.4.1 yields the desired result. □

The smoothness of a $J\cap\mathfrak{p}$ can be deduced from a general result that we now recall, see, for example, [Iv, Proposition 1.3] or [PV, 6.5, Corollary].

Theorem 2.4.5. *Let Γ be a reductive group acting on a smooth variety X. Then the subvariety of fixed points $X^{\Gamma}=\{x\in X\mid\Gamma.x=x\}$ is smooth, and $T_xX^{\Gamma}=(T_xX)^{\Gamma}$ for all $x\in X^{\Gamma}$.*

This theorem can be applied to a J_G-class J as follows. Let

$$\Gamma=\{\mathrm{Id},\tilde{\theta}\}\subset\mathrm{GL}(\mathfrak{g})$$

be the group, of order two, generated by $\tilde{\theta}=-\theta$ (thus $\tilde{\theta}$ is an anti-automorphism of $\mathfrak{g}$). Now, we can note [TY, 39.1.7] that $J=J_G(x)=G.\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^x)^{\bullet}$. From the definition of a Jordan class, or this description, it follows that J is stable under the $\Bbbk^{\times}$-action $y\mapsto\lambda y$, $\lambda\in\Bbbk^{\times}$ so, when $J\cap\mathfrak{p}\neq\emptyset$, we have $\tilde{\theta}(J)=\theta(J)=J$. Therefore, the group Γ acts on the smooth variety J and we get from Theorem 2.4.5 that $J^{\Gamma}=J\cap\mathfrak{p}$ is smooth. This provides another proof of Theorem 2.4.4 (see the four last lines in the proof of that theorem).

2.5 K-sheets

We continue with the same notation. Fix a G-sheet $S=S_G\subset\mathfrak{g}^{(2m)}$, $m\in\mathbb{N}$. We aim to describe the irreducible components of $S_G\cap\mathfrak{p}$. One important remark is that if $S_G\cap\mathfrak{p}\neq\emptyset$, then the unique nilpotent orbit $\mathcal{O}$ contained in S_G intersects $\mathfrak{p}$ (cf. [TY, 39.6.2]). The description of the irreducible components of $S_G\cap\mathfrak{p}$ will be given in terms of the K-orbits contained in $\mathcal{O}$, see Theorem 2.5.11.

We first want to prove that when S is smooth, and $(\mathfrak{g},\theta)$ has no irreducible factor of type 0, the intersection $S\cap\mathfrak{p}$ (which can be empty) is also smooth. To obtain this result we will apply Theorem 2.4.5, as in the case of a Jordan G-class. We adopt the notation of the end of the previous subsection, in particular we set $\Gamma=\{\mathrm{Id},\tilde{\theta}=-\theta\}$. Observe that S is stable under the $\Bbbk^{\times}$-action, thus $\tilde{\theta}(S)=\theta(S)$; but, contrary to the case of a Jordan class, the stability of S under Γ requires some hypothesis, even in the case where $S\cap\mathfrak{p}\neq\emptyset$.

We begin with the following, probably known, technical result. Recall [CM, 7.1] that a nilpotent orbit $\mathcal{O}$ is called *rigid* if it can not be obtained by induction of a proper parabolic subalgebra of $\mathfrak{g}$; equivalently, when $\mathfrak{g}$ is semisimple, $\mathcal{O}$ is rigid if $\mathcal{O}$ is a G-sheet, cf. [Boh, §4]. Recall also that the only rigid orbit in type A is $\{0\}$, see [Kr, 2.4] or [CM].

Lemma 2.5.1. *Let $\mathfrak{l}$ be a Levi factor of a simple Lie algebra $\mathfrak{g}$ and $\mathcal{O}$ be a rigid nilpotent orbit of $\mathfrak{l}$. Then, $\tau(\mathcal{O}) = \mathcal{O}$ for all $\tau \in \mathrm{Aut}(\mathfrak{l})$.*

Proof. Observe first that $\tau(\mathcal{O})$ is a rigid nilpotent orbit. Decompose $\mathfrak{l}$ as the direct sum of its center and simple factors, $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \bigoplus_j \mathfrak{l}_j$. Write $\mathcal{O} = \prod_j \mathcal{O}_j$ where $\mathcal{O}_j \subset \mathfrak{l}_j$ is a nilpotent orbit. It is easy to see that the orbits $\mathcal{O}_j$ are rigid; observe also that if each $\mathfrak{l}_j$ is of type A, $\mathcal{O}$ is zero and, obviously, $\tau(\mathcal{O}) = \mathcal{O}$. From the classification of Dynkin diagrams one deduces that for each subdiagram of the Dynkin diagram of $\mathfrak{g}$, there exists at most one connected component of type different from A. Therefore, we are reduced to the case when there exists an index j such that $\mathfrak{l}_j$ is not of type A. By uniqueness of j one has $\tau(\mathfrak{l}_j) = \mathfrak{l}_j$. Set $\mathfrak{m} = \mathfrak{l}_j$ and $\Omega = \mathcal{O}_j$; by the previous remarks it remains to show that $\Omega = \tau(\Omega) \subset \mathfrak{m}$.
Recall that $\mathrm{Aut}(\mathfrak{m})$ is the semidirect product of the adjoint group M of $\mathfrak{m}$ and of the group $\mathrm{Out}(\mathfrak{m})$, isomorphic to the group of automorphisms of its Dynkin diagram, cf. [Bou, Ch. VIII, §5.3].
– If $\mathfrak{m}$ is of type B_n, C_n, E_7, E_8, F_4 or G_2, then $\mathrm{Aut}(\mathfrak{m}) = M$ and $\tau(\Omega) = \Omega$ is clear.
– Suppose that $\mathfrak{m}$ is of type E_6; then, $\mathrm{Out}(\mathfrak{m}) = \{\mathrm{Id}, \omega\}$ has order two. Recall that weighted Dynkin diagrams are in one to one correspondence with nilpotent orbits, see [CM]. Then, going through the list of weighted Dynkin diagrams in type E_6 (cf. [CM, 8.4]), one sees that each of these diagrams is fixed by the automorphism induced by ω. Thus, $\omega(\Omega) = \Omega$ and it follows that $\tau(\Omega) = \Omega$ for all $\tau \in \mathrm{Aut}(\mathfrak{m})$.
– Assume that $\mathfrak{m} \cong \mathfrak{so}_{2n}$ is of type D_n with $n \geqslant 5$; here, as in the previous case, $\#\,\mathrm{Out}(\mathfrak{m}) = 2$. Let $\boldsymbol{\lambda}$ be the partition of $2n$ associated to Ω. From the classification of rigid orbits one deduces that $\boldsymbol{\lambda}$ is not very even and it follows that Ω is stable under $\mathrm{Aut}(\mathfrak{m})$ (see [CM, 7.3] for these assertions).
– The last remaining case is when $\mathfrak{m}$ is of type D_4. By [CM, 7.3] (for example), one gets that there exist exactly two nonzero rigid orbits in $\mathfrak{m}$, which have different dimensions. Thus Ω is stable under $\mathrm{Aut}(\mathfrak{m})$. □

The next lemma ensures that when $\mathfrak{g}$ is simple, the smoothness of S is inherited by $S \cup \theta(S)$.

Lemma 2.5.2. *Let $\mathfrak{g}$ be a simple Lie algebra. If $\mathfrak{g}$ is not of type* D, *then $\theta(S) = S$.*
If $\mathfrak{g}$ is of type D, *one has either $\theta(S) = S$ or $S \cap \theta(S) = \emptyset$.*

Proof. Let J_1 be the dense Jordan class contained in S and let $(\mathfrak{l}, \mathcal{O})$ be a datum of J_1. Then, the dense Jordan class J_2 in the sheet $\theta(S)$ has datum $(\theta(\mathfrak{l}), \theta(\mathcal{O}))$.
If $\mathfrak{g}$ is of type different from D or E_7, it follows from the classification of Levi factors in [Dy, Theorem 5.4] that $\theta(\mathfrak{l})$ is G-conjugate to $\mathfrak{l}$ (cf. also [BC, Proposition 6.3]). In these cases we can

therefore assume that $\theta(\mathfrak{l}) = \mathfrak{l}$, and Lemma 2.5.1 yields $\theta(\mathcal{O}) = \mathcal{O}$. Thus, $J_1 = J_2$ and $\theta(S) = S$. If $\mathfrak{g}$ is of type E_7, there exists no outer automorphism of $\mathfrak{g}$ so $\theta(S) \subseteq G.S = S$.
Suppose that $\mathfrak{g}$ is of type D. If $\mathfrak{l}$ and $\theta(\mathfrak{l})$ are G-conjugate, the previous argument applies and one gets $\theta(S) = S$. Otherwise, [IH, Corollary 3.15] implies that $S \cap \theta(S) = \overline{J_1}^{\bullet} \cap \overline{J_2}^{\bullet} = \emptyset$. $\square$

We can now prove the desired result:

Proposition 2.5.3. (i) *Let $(\mathfrak{g}, \theta)$ be a reductive symmetric Lie algebra which has no irreducible factor of type* 0. *If S is a smooth G-sheet, the intersection $S \cap \mathfrak{p}$ is smooth.*
(ii) *Let $(\mathfrak{g}, \theta)$ be a symmetric Lie algebra and S' be a K-sheet contained in a smooth G-sheet S. Then S' is smooth.*

Proof. Decompose the symmetric algebra $(\mathfrak{g}, \theta)$ as $(\mathfrak{z}(\mathfrak{g}), \theta_{|\mathfrak{z}(\mathfrak{g})}) \oplus \bigoplus_i (\mathfrak{g}_i, \theta_{|\mathfrak{g}_i})$ where each $(\mathfrak{g}_i, \theta_{|\mathfrak{g}_i})$ is an irreducible factor (see the beginning of this section).
(i) We want to apply Theorem 2.4.5 with $\Gamma = \{\mathrm{Id}, \tilde{\theta} = -\theta\}$ and $X = S \cup \theta(S) \subset \mathfrak{g}$. Note that $X^{\Gamma} = (S \cap \mathfrak{p}) \cup (\theta(S) \cap \mathfrak{p})$ and that $\theta(S)$ is smooth.
If $\mathfrak{g}$ is simple, Lemma 2.5.2 yields that $X = S$ or $S \sqcup \theta(S)$ (in type D) is smooth; therefore X^{Γ}, and consequently $S \cap \mathfrak{p}$, is smooth. Suppose that $\mathfrak{g}$ is not simple. By hypothesis, each $\mathfrak{g}_i$ is simple and the result then follows from Corollary 1.7.2.
(ii) By a similar reduction to the irreducible factors $(\mathfrak{g}_i, \theta_{|\mathfrak{g}_i})$ the only remaining case to consider is that of type 0, i.e., $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$ with $\theta : \mathfrak{g}^1 \xrightarrow{\sim} \mathfrak{g}^2$. From the results of §2.1 it follows that there exists a G^1-sheet $S^1 \subset \mathfrak{g}^1$ such that $S' = \{x - \theta(x) \mid x \in S^1\}$. Then $S = S^1 \times \theta(S^1)$, which is smooth if and only if S^1 is smooth. As S^1 is isomorphic to S', one gets the desired result. $\square$

Remarks 2.5.4. (1) The sheets in a classical Lie algebra are smooth, see Theorem 1.4.10. Therefore if $\mathfrak{g}$ is simple of classical type, Proposition 2.5.3 implies that $S \cap \mathfrak{p}$ is smooth for each sheet S of $\mathfrak{g}$.
(2) When $\mathfrak{g} = \mathfrak{gl}_N$, case which will be studied in details in Section 3, the smoothness of $S_G \cap \mathfrak{p}$ can been explained in different (equivalent) terms. Indeed, recall first that, if $\mathfrak{g} = \mathfrak{gl}_N$, a nilpotent orbit is contained in a unique G-sheet, cf. Remark 1.4.3. Assume that the sheet $S = S_G$ intersects $\mathfrak{p}$ and let $\mathcal{O} = G.e$ be the nilpotent orbit contained in S. Then, since we may assume that $e \in \mathfrak{p}$, it follows from $G.\theta(e) = G.(-e) = G.e \subset \theta(S) \cap S$ that $\theta(S) = S$. Therefore, the group Γ acts on S and $S^{\Gamma} = S \cap \mathfrak{p}$ is smooth.

Assume that the sheet S_G intersects $\mathfrak{p}$, pick $e \in \mathcal{O} \cap \mathfrak{p}$ and set

$$\mathcal{O}_e = K.e \subset \mathcal{O} \cap \mathfrak{p}.$$

Denote by $\mathscr{S} = (e, h, f)$ a normal $\mathfrak{sl}_2$-triple containing e. We are going to apply the results recalled in §1.4 to various triples deduced from $\mathscr{S}$. Recall from Remarks 1.4.11 that these results hold for any such $\mathfrak{sl}_2$-triple.
Let $\mathsf{Z} \subset G$ be a subset such that $\{g.e\}_{g \in \mathsf{Z}}$ is a set of representatives of the K-orbits contained in

$\mathcal{O} \cap \mathfrak{p}$; we assume that $\mathrm{Id} \in \mathsf{Z}$. Observe that, since the $\mathfrak{sl}_2$-triples containing $g.e$ are conjugate, we may also assume that $g.\mathscr{S} := (g.e, g.h, g.f)$ is a normal $\mathfrak{sl}_2$-triple for all $g \in \mathsf{Z}$. Recall that $X(S_G, g.\mathscr{S})$ is defined by

$$g.e + X(S_G, g.\mathscr{S}) = S_G \cap (g.e + \mathfrak{g}^{g.f}) = g.(S_G \cap (e + \mathfrak{g}^f)) = g.(e + X(S_G, \mathscr{S})).$$

(Hence $X(S_G, g.\mathscr{S}) = g.X(S_G, \mathscr{S})$.) Set

$$X_{\mathfrak{p}}(S_G, g.\mathscr{S}) = X(S_G, g.\mathscr{S}) \cap \mathfrak{p}. \tag{2.4}$$

Remark 2.5.5. Recall that $S \subset \mathfrak{g}^{(2m)}$. Let $\emptyset \neq Y \subset g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})$; then, each G-orbit (resp. K-orbit) of an element of Y has dimension $\dim G.e = 2m$ (resp. $\dim K.e = m$). Lemma 1.4.6 implies that the fibers of the morphisms $G \times Y \to G.Y$ and $K \times Y \to K.Y$ are of respective dimension $\dim G^e$ and $\dim K^e$. Then, by [TY, 15.5.5], we get that $\dim G.Y = \dim Y + 2m$ and $\dim K.Y = \dim Y + m$.

We now introduce some conditions which will be sufficient to give a description of the irreducible components of $S_G \cap \mathfrak{p}$ in terms of the $X_{\mathfrak{p}}(S_G, g.\mathscr{S})$, see Theorem 2.5.11.

Recall that $S_G = G.(e + X(S_G, \mathscr{S}))$. The first condition ensures that $e + X_{\mathfrak{p}}$ is large enough:

$$G.(g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})) = G.(S_G \cap \mathfrak{p}) \text{ for all } g \in \mathsf{Z}. \tag{$\heartsuit$}$$

The condition ($\heartsuit$) was established for pairs of type 0 in Proposition 2.1.4, and we will see that it also holds for all symmetric pairs when $\mathfrak{g} = \mathfrak{gl}_N$ (cf. Theorem 3.2.1). Set:

$$A(g.e) = G^{g.e}/(G^{g.e})^{\circ}.$$

By Theorem 1.4.7 the Slodowy slice $g.e + X(S_G, g.\mathscr{S})$ provides the geometric quotient

$$\psi_{S_G, g.\mathscr{S}} : S_G \longrightarrow (g.e + X(S_G, g.\mathscr{S}))/A(g.e)$$

and we will be interested in some cases where the following property is satisfied:

$$G^e \text{ is connected.} \tag{$*$}$$

Recall that ($*$) is true when $\mathfrak{g} = \mathfrak{gl}_N$ (see Lemma 1.6.1). Clearly, ($*$) implies that $g.e + X(S_G, g.\mathscr{S})$ is the geometric quotient of S_G. In this case, the restriction of $\psi_{S_G, g.\mathscr{S}}$ to the subset $(g.e + \bigoplus_{i \leqslant 0} \mathfrak{g}(2i, g.h)) \cap S_G$ is given by the map $\varepsilon_{S_G, g.\mathscr{S}}$ constructed in Lemma 1.4.4, and if hypothesis ($\heartsuit$) is also satisfied, one has: $\psi_{S_G, g.\mathscr{S}}(S_G \cap \mathfrak{p}) = g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})$.

Let J_1 be a J_K-class contained in $S_G \cap \mathfrak{p}$. As J_1 is K-stable, the dimension of $J_1 \cap (g.e + \mathfrak{p}^{g.f})$ does not depend on the representative element $g.\mathscr{S}$ of the orbit $K.g.\mathscr{S}$. Since K-orbits of normal $\mathfrak{sl}_2$-triples are in one to one correspondence with K-orbits of their nilpositive part (*i.e.* the first element of such an $\mathfrak{sl}_2$-triple), we may introduce the following definition.

Definition 2.5.6. Let $g \in \mathsf{Z}$. A J_K-class J_1 contained in S_G is said to be *well-behaved with respect to* $\mathcal{O}_{g.e} = K.g.e$, if:

$$\dim J_1 \cap (g.e + \mathfrak{p}^{g.f}) = \dim J_1 - m. \tag{2.5}$$

Remark 2.5.7. It follows from Lemma 2.4.2(ii) that a J_K-class $J_1 = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n)$ is well-behaved w.r.t. $\mathcal{O}_{g.e}$ if and only if $Y = J_1 \cap (g.e + \mathfrak{p}^{g.f})$ satisfies $\dim Y = \dim \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)$ $(= \dim J_1 - m)$. By Remark 2.5.5 this is also equivalent to $\dim K.Y = \dim J_1$, which is in turn equivalent to $J_1 \subset \overline{K.Y}$. In this case one has $J_1 \subset \overline{K.(g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S}))}$, property which will be of importance for the description of $S_G \cap \mathfrak{p}$.

The following lemma shows that, assuming ($\heartsuit$), well-behaved J_K-classes exist.

Lemma 2.5.8. *Let J be a J_G-class contained in S_G such that $J \cap \mathfrak{p} \neq \emptyset$. Fix $g \in \mathsf{Z}$ and set $\psi = \psi_{S_G, g.\mathscr{S}}$. Assume that the property ($\heartsuit$) is satisfied.*
(i) *Let $J_1 \subset J \cap \mathfrak{p}$ be a J_K-class. There exists a subvariety $Y \subset g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})$ such that: Y is irreducible and $\psi(Y)$ is dense in $\psi(J_1)$. Moreover, if $Y \subset g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})$ is maximal for these two properties, then $\psi(Y) = \psi(J_1)$ and $J_2 = \overline{K.Y} \cap J$ is a J_K-class (contained in J) which is well-behaved w.r.t. $\mathcal{O}_{g.e}$.*
(ii) *The class J_1 is well-behaved w.r.t. $\mathcal{O}_{g.e}$ if and only if one can find Y, as in* (i)*, such that $J_1 = \overline{K.Y} \cap J$.*
(iii) *If (*) holds, there exists a unique maximal Y as in* (i)*, namely $Y = \psi_{S_G, g.\mathscr{S}}(J_1)$, thus $J_2 = \overline{K.\psi_{S_G, g.\mathscr{S}}(J_1)} \cap J$.*

Proof. In order to simplify the notation, we suppose that $g = \mathrm{Id}$ and we set $X = X(S_G, \mathscr{S})$, $X_{\mathfrak{p}} = X \cap \mathfrak{p}$, $\psi = \psi_{S_G, \mathscr{S}}$, $S = S_G$, etc.
(i) Consider the following commutative diagram

$$\begin{tikzcd} e + X_{\mathfrak{p}} \arrow[r, "i"] \arrow[dr, "\gamma_{\mathfrak{p}}"'] & e + X \arrow[d, "\gamma"] \\ & (e + X)/A = \psi(S) \end{tikzcd}$$

where i is the natural closed embedding and γ is the quotient morphism, see (1.5). Observe that, the group A being finite, the morphisms γ and $\gamma_{\mathfrak{p}}$ are finite, hence closed. Moreover, ($\heartsuit$) implies that $\mathrm{im}(\gamma_{\mathfrak{p}}) = \psi(S \cap \mathfrak{p})$. Let Y' be any irreducible component of $\gamma_{\mathfrak{p}}^{-1}(\overline{\psi(J_1)})$ dominating $\overline{\psi(J_1)} \subset (e + X)/A$ and set:

$$Y = \gamma_{\mathfrak{p}}^{-1}(\psi(J_1)) \cap Y'.$$

Then, $Y \subset J$ is a dense irreducible subset of Y' such that $\psi(Y) = \gamma_{\mathfrak{p}}(Y) = \psi(J_1)$. Since the fibers of ψ are of dimension m and $\gamma_{\mathfrak{p}}$ is finite, one has $\dim Y = \dim J_1 - m$. Set

$$J_2 = \overline{K.Y} \cap J.$$

As $K.Y \subset J_2 \subset J \cap \mathfrak{p}$, J_2 is a closed irreducible subset of $J \cap \mathfrak{p}$ of dimension $\dim K.Y = \dim Y + m = \dim J \cap \mathfrak{p}$ (cf. Remark 2.5.5). One obtains from Theorem 2.4.4 that J_2 is a J_K-class, which is well-behaved w.r.t. $\mathcal{O}_e$ (recall that $J_2 \subset \overline{K.Y}$).
Suppose now that $Y_1 \subset e + X_{\mathfrak{p}}$ is maximal for the properties: Y_1 irreducible and $\psi(Y_1)$ dense in $\psi(J_1)$. Observe that the closure Y_1' of Y_1 inside $e + X_{\mathfrak{p}}$ is irreducible, and $\gamma_{\mathfrak{p}}(Y_1') = \overline{\gamma_{\mathfrak{p}}(Y_1)} = \overline{\psi(J_1)}$. The argument of the previous paragraph, together with the maximality of Y_1, implies that $Y_1 = \gamma_{\mathfrak{p}}^{-1}(\psi(J_1)) \cap Y_1'$. As above, we then get that $\overline{K.Y_1} \cap J$ is a well-behaved J_K-class contained in $J \cap \mathfrak{p}$.
(ii) Set $Y_1 = J_1 \cap (e + X_{\mathfrak{p}})$ and suppose that J_1 is well-behaved w.r.t. $\mathcal{O}_e$, thus $\dim J_1 = \dim Y_1 + m$. Let $Y_2 \subset Y_1$ be an irreducible component of maximal dimension; since $\gamma_{\mathfrak{p}}$ is finite, one has $\dim \gamma_{\mathfrak{p}}(Y_2) = \dim Y_1 = \dim \psi(J_1)$, hence $\psi(Y_2)$ is dense in $\psi(J_1)$. We then deduce from (i) that $J_2 = \overline{K.Y_2} \cap J$ is a J_K-class; since $Y_2 \subset J_2 \cap J_1$, it follows that $J_1 = J_2$ is well-behaved w.r.t. $\mathcal{O}_e$. The converse is clear.
(iii) Here, $\gamma_{\mathfrak{p}} : e + X_{\mathfrak{p}} \xrightarrow{\sim} \psi(S \cap \mathfrak{p})$ is the identity; thus $Y' = \overline{\psi(J_1)}$ and $Y = \psi(J_1)$. □

Remarks 2.5.9. (1) In part (i) of the previous lemma, the J_K-class J_2 ($\subset J \subset S_G$) is contained in the following variety:

$$S_K(S_G, g.\mathscr{S}) := \overline{K.(g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S}))}^{\bullet}. \tag{2.6}$$

Since K-orbits of normal $\mathfrak{sl}_2$-triples are in bijection with nilpotent K-orbits, $S_K(S_G, g.\mathscr{S})$ depends only on the sheet S_G and the orbit $\mathcal{O}_{g.e} = K.g.e$. Therefore we can write

$$S_K(S_G, g.\mathscr{S}) = S_K(S_G, \mathcal{O}_{g.e}).$$

Furthermore when $\mathfrak{g}$ is of type A, thanks to Remark 1.4.3, we may also write $S_K(S_G, g.\mathscr{S}) = S_K(g.\mathscr{S}) = S_K(\mathcal{O}_{g.e})$.
(2) Under assumption (*), Lemma 2.5.8(iii) yields a well defined application

$$J_1 \mapsto J_2 = J \cap \overline{K.\psi_{S_G, g.\mathscr{S}}(J_1)}$$

from the set of J_K-classes contained in $S_G \cap \mathfrak{p}$ to the set of J_K-classes contained in $S_K(S_G, g.\mathcal{O})$. In case A, we will show in Lemma 3.3.5 and Lemma 3.3.11 that each J_K-class contained in $S_G \cap \mathfrak{p}$ is in the image of such an application, for an appropriate choice of $g \in \mathsf{Z}$.

We now introduce a condition ensuring that the varieties $S_K(S_G, \mathcal{O}_e)$ are irreducible:

$$X_{\mathfrak{p}}(S_G, g.\mathscr{S}) \text{ is irreducible for all } g \in \mathsf{Z}. \tag{$\diamondsuit$}$$

Corollary 2.5.10. *Assume that conditions ($\heartsuit$) and ($\diamondsuit$) hold. Then, $S_K(S_G, \mathcal{O}_{g.e})$ is an irreducible component of $S_G \cap \mathfrak{p}$ of maximal dimension.*

Proof. Let J_1 be a J_K-class of maximal dimension contained in $S_G \cap \mathfrak{p}$ and $J \subset S_G$ be the J_G-class containing J_1. Since ($\heartsuit$) is satisfied, one can find Y as in Lemma 2.5.8(i) such that $J_2 = \overline{K.Y} \cap J$

is a J_K-class contained in J. Then, $J_2 \subset S_K(S_G, \mathcal{O}_{g.e}) \subset S_G \cap \mathfrak{p}$ and Theorem 2.4.4 implies that $\dim J_2 = \dim J_1 = \dim S_G \cap \mathfrak{p}$. Therefore $S_K(S_G, \mathcal{O}_{g.e}) = \overline{J_2}^{\bullet}$ is an irreducible component of $S_G \cap \mathfrak{p}$ of maximal dimension. □

In view of the previous corollary, it is then natural to ask: Are all the irreducible components of $S_G \cap \mathfrak{p}$ of the form $S_K(S_G, \mathcal{O}_{g.e})$? We introduce the next additional condition to answer that question:

For each J_K-class J_1 in $S_G \cap \mathfrak{p}$, there exists $g \in \mathsf{Z}$ such that J_1 is well-behaved w.r.t. $\mathcal{O}_{g.e}$. (♣)

Theorem 2.5.11. *Under conditions (♡), (♢) and (♣), the irreducible components of $S_G \cap \mathfrak{p}$ are the $S_K(S_G, \mathcal{O}_{g.e})$ with $g \in \mathsf{Z}$. Consequently, $S_G \cap \mathfrak{p}$ is equidimensional. Moreover, there exists a unique J_G-class J such that $S_G \cap \mathfrak{p} = \overline{J \cap \mathfrak{p}}^{\bullet}$ and, for each $g \in \mathsf{Z}$, $S_K(S_G, \mathcal{O}_{g.e}) = \overline{J_g}^{\bullet}$ for a unique J_K-class J_g. In particular, the map $S_K(S_G, \mathcal{O}_{g.e}) \to J_g$ gives a bijection between irreducible components of $S_G \cap \mathfrak{p}$ and the set of J_K-classes contained in $J \cap \mathfrak{p}$.*

Proof. Write $S_G \cap \mathfrak{p} = \bigcup_{J \subset S_G} J \cap \mathfrak{p}$, where the union is taken over the J_G-classes J intersecting $\mathfrak{p}$. For any such J, $J \cap \mathfrak{p}$ is the union of the J_K-classes it contains (cf. Lemma 2.4.1), thus (♣) and Lemma 2.5.8(ii) imply that $S_G \cap \mathfrak{p} = \bigcup_{g \in \mathsf{Z}} S_K(S_G, \mathcal{O}_{g.e})$. Since (♡) and (♢) are satisfied, one may apply Corollary 2.5.10 to get the two first claims.
Now, let J_1 be a J_K-class of maximal dimension contained in $S_G \cap \mathfrak{p}$ and denote by $J \subset S_G$ the J_G-class containing J_1. Let $g \in \mathsf{Z}$; as in the proof of Corollary 2.5.10 one can find a J_K-class $J_g \subset J \cap \mathfrak{p}$ such that $S_K(S_G, \mathcal{O}_{g.e}) = \overline{J_g}^{\bullet}$. It then follows from the previous paragraph that $S_G \cap \mathfrak{p} = \overline{J \cap \mathfrak{p}}^{\bullet}$. Furthermore, as J_K-classes are locally closed, J_g is the unique dense J_K-class in $S_K(S_G, \mathcal{O}_{g.e})$. This implies the unicity of the class J. Finally, the bijection follows from the first claim and Theorem 2.4.4. □

We have observed in Remark 2.5.4 that when $\mathfrak{g}$ is a simple classical Lie algebra, the variety $S_G \cap \mathfrak{p}$ is smooth. We are going to introduce two new conditions in order to obtain the smoothness of $S_G \cap \mathfrak{p}$ in the general case.

Recall from Lemma 1.4.6(ii) that the G-action induces an action of the group $A(g.e) \cong G^e/(G^e)^\circ$ on $g.e + X(S_G, g.\mathscr{S})$. We consider the following condition:

$$A(g.e).(g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})) \text{ is a smooth variety for all } g \in \mathsf{Z}. \tag{2.7}$$

Remarks 2.5.12. (1) Set $X_{\mathfrak{p}} = X_{\mathfrak{p}}(S_G, g.\mathscr{S})$ and $A = A(g.e)$. Using the $(F_t)_{t \in \Bbbk^\times}$-action (see §1.4), one can see that $g.e$ belongs to each irreducible component of $g.e + X_{\mathfrak{p}}$. Suppose that (2.7) is satisfied. Since A is finite, this implies that, for all $a \in A$, $g.e + X_{\mathfrak{p}} = a.(g.e + X_{\mathfrak{p}}) = A.(g.e + X_{\mathfrak{p}})$ is a smooth irreducible variety. In particular, $X_{\mathfrak{p}}$ is smooth and irreducible, and the condition (2.7) is stronger than (♢).
(2) Using an analogue of Proposition 1.4.9 it is possible to show that the smoothness of $S_G \cap \mathfrak{p}$

implies the smoothness of $X_{\mathfrak{p}}(S_G, g.\mathscr{S})$, and that $K.(g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S}))$ is then also smooth. But the latter variety is in general not an irreducible component of $S_G \cap \mathfrak{p}$, even for some cases in types AI or AIII.

The last condition we want to consider is:

$$\text{For each } J_K\text{-class } J_1 \subset S_G \cap \mathfrak{p} \text{ and all } x, y \in J_1, \text{ one has: } G.x = G.y \Leftrightarrow K.x = K.y. \tag{2.8}$$

Even if conditions (2.7) and (2.8) will not be used in the sequel of the paper, it is worth noticing that they have interesting consequences on the smoothness of the intersection $S_G \cap \mathfrak{p}$, see Proposition 2.5.14. Some technical arguments, combined with results from section 3, can show that these consditions hold when $\mathfrak{g} = \mathfrak{gl}_N$.
We first remark that the conditions (♡) and (2.8) yield a slight improvement of Lemma 2.5.8.

Lemma 2.5.13. *Let J be a J_G-class contained in S_G such that $J \cap \mathfrak{p} \neq \emptyset$ and adopt the notation of* Lemma 2.5.8(i)*. Assume that the properties* (♡) *and* (2.8) *hold. Then the class $J_2 \subset J$ satisfies $J_2 = K.Y$.*

Proof. Under the notation of the proof of Lemma 2.5.8(i), choose $x \in Y$ such that $x \in G.J_1$. By Lemma 2.4.3(ii) one gets $G.J_1 = G.J_2$ and $\psi(J_1) = \psi(J_2)$. It follows from $\gamma_{\mathfrak{p}}(Y) = \psi(J_2)$ and (2.8) that $J_2 = K.Y$. □

We can now summarize in the following result the consequences of the four conditions previously introduced:

Proposition 2.5.14. *Under the four conditions* (♡), (♣), (2.7), (2.8)*, the variety $S_G \cap \mathfrak{p}$ is smooth and its irreducible components are of the form $S_K(S_G, \mathcal{O}_{g.e})$ with $g \in \mathsf{Z}$.*

Proof. Set $S = S_G$ and recall that $S' = G.(S \cap \mathfrak{p}) = G.(g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S}))$ for all $g \in \mathsf{Z}$, cf. (♡). Let $J_1 \subset S \cap \mathfrak{p}$ be a J_K-class. By (♣), J_1 is well-behaved w.r.t. $\mathcal{O}_{g.e}$ for some $g \in \mathsf{Z}$. Then, Lemma 2.5.8(ii) and Lemma 2.5.13 imply $J_1 = K.Y$ with $Y \subset g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S}) \subset S \cap \mathfrak{p}$. It follows that:

$$S \cap \mathfrak{p} = S' \cap \mathfrak{p} = \bigcup_{g \in \mathsf{Z}} K.(g.e + X_{\mathfrak{p}}(S_G, g.\mathscr{S})). \tag{2.9}$$

Fix $g \in \mathsf{Z}$ and set $X_{\mathfrak{p}} = X_{\mathfrak{p}}(S_G, g.\mathscr{S})$. From (♡) and Lemma 1.4.6(iii) we deduce that $S' \cap (g.e + \mathfrak{g}^{g.f}) = A(g.e).(g.e + X_{\mathfrak{p}})$. Since $S' = G.(g.e + X_{\mathfrak{p}})$, Proposition 1.4.9(iii) gives that S' is smooth if and only if $S' \cap (g.e + \mathfrak{g}^{g.f}) = A(g.e).(g.e + X_{\mathfrak{p}})$ is smooth. Hence, by (2.7), S' is smooth.
We now compute the tangent space to $S' \cap \mathfrak{p}$ at a point $g.e + x \in g.e + X_{\mathfrak{p}}$. Using Remark 2.5.12(1) one gets that $S' \cap (g.e + \mathfrak{g}^{g.f}) = g.e + X_{\mathfrak{p}}$ and Proposition 1.4.9, again, yields:

$$T_{g.e+x} S' = [\mathfrak{g}, x] \oplus T_x X_{\mathfrak{p}} = [\mathfrak{k}, x] \oplus [\mathfrak{p}, x] \oplus T_x X_{\mathfrak{p}}.$$

It follows that $T_{g.e+x}(S' \cap \mathfrak{p}) \subseteq T_{g.e+x}S' \cap \mathfrak{p} = [\mathfrak{k}, x] \oplus T_x X_{\mathfrak{p}}$. On the other hand, by (2.9), $T_{g.e+x}(S' \cap \mathfrak{p}) \supseteq T_{g.e+x}K.(g.e + X_{\mathfrak{p}}) \supseteq [\mathfrak{k}, x] \oplus T_x X_{\mathfrak{p}}$. Thus:

$$T_{g.e+x}(S' \cap \mathfrak{p}) = [\mathfrak{k}, x] \oplus T_x X_{\mathfrak{p}}.$$

Since $X_{\mathfrak{p}}$ is smooth, we obtain: $\dim T_{g.e+x}(S' \cap \mathfrak{p}) = \dim K.x + \dim T_x X_{\mathfrak{p}} = m + \dim X_{\mathfrak{p}}$. Therefore, each element of $g.e + X_{\mathfrak{p}}$ is a smooth point of $S' \cap \mathfrak{p} = S \cap \mathfrak{p}$, and (2.9) then implies that $S \cap \mathfrak{p}$ is smooth. The last assertion is given by Theorem 2.5.11. $\square$

3 Type A

We show in this section that the conditions (♡), (♢) and (♣), introduced in Section 2.5 in order to describe the K-sheets of a reductive (or semisimple, see Corollary 1.7.2) symmetric Lie algebra $(\mathfrak{g}, \theta)$, are satisfied in type A, i.e. when $\mathfrak{g} = \mathfrak{gl}_N$ (or $\mathfrak{sl}_N$).

Thereafter, unless otherwise specified, e.g. in 3.1.1, we set $\mathfrak{g} = \mathfrak{gl}_N$, $N \in \mathbb{N}^*$, and if θ is an involution on $\mathfrak{g}$ we adopt the notation of Section 2 relative to the symmetric pair $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k})$. The natural action of $\tilde{G} = \mathrm{GL}_N$ on $\mathfrak{g}$ factorizes through the adjoint action to give the surjective morphism:

$$\rho : \tilde{G} \longrightarrow G \cong \tilde{G}/\Bbbk^{\times}\mathrm{Id} = \mathrm{PGL}_N = \mathrm{PSL}_N$$

Recall that $G^{\theta} = \{g \in G \mid g \circ \theta = \theta \circ g\}$ and $K = (G^{\theta})^{\circ}$. If H is an algebraic subgroup of G we set:

$$\tilde{H} = \rho^{-1}(H). \tag{3.1}$$

Thus, $H.x = \tilde{H}.x$ for all $x \in \mathfrak{g}$. After recalling the three different possible types of involutions, we will establish the three aforementioned conditions:

(♡) in Theorem 3.2.1 (types AI, AII) and Proposition 3.2.6 (type AIII);

(♢) in Remark 3.2.3 (types AI, AII) and Remark 3.2.8 (type AIII);

(♣) in Corollary 3.3.5 (types AI, AII) and Proposition 3.3.11 (type AIII).

3.1 Involutions in type A

We recall below a construction of the involutions on $\mathfrak{gl}_N = \mathfrak{gl}(V)$. We will also have to consider the involution by permutation of factors on $\mathfrak{gl}_N \times \mathfrak{gl}_N$, cf. 2.1; this case will be called "type A0".

Recall that the nilpotent orbits in $\mathfrak{g} = \mathfrak{gl}_N$ are in bijection with the partitions of N and that, to each partition $\boldsymbol{\mu} = (\mu_1 \geqslant \cdots \geqslant \mu_k)$, one associates a Young diagram having μ_i boxes on the i-th row.

We fix a G-sheet $S_G \subset \mathfrak{g}$ and an element e in the nilpotent orbit $\mathcal{O} \subset S_G$. The partition associated to e is denoted by

$$\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\delta_{\mathcal{O}}}).$$

We adopt the notation introduced in 1.6.1; in particular, the basis $\mathbf{v}$ (in which $e = \sum_i e_i$ has a Jordan normal form, see (1.6)) and the subalgebras $\mathfrak{q}_i \cong \mathfrak{gl}_{\lambda_i}$, $\mathfrak{q} = \oplus_i \mathfrak{q}_i, \mathfrak{l}, \mathfrak{t}$ are fixed.

We want to construct symmetric pairs $(\mathfrak{g},\theta)\equiv(\mathfrak{g},\mathfrak{k},\mathfrak{p})\equiv(\mathfrak{g},\mathfrak{k})$ such that $e\in\mathfrak{p}$. These constructions are inspired from [Oh1, Oh2]. The notation being as in [He1, GW], one obtains three types of non-isomorphic symmetric pairs: AI, AII and AIII. Recall that the involution θ is outer in types AI, AII and inner in type AIII.
The most complicated case is type AIII, where it is possible to embed e in several non-isomorphic ways in different $\mathfrak{p}$'s. These possibilities will be parameterized by functions $\Phi:[\![1,\delta_{\mathcal{O}}]\!]\to\{a,b\}$, where a,b are different symbols.

3.1.1 Case A0

Let θ be the involution on $\mathfrak{g}=\mathfrak{gl}_N\times\mathfrak{gl}_N$ sending (x,y) to (y,x). Recall that $\mathfrak{k}=\{(x,x)\mid x\in\mathfrak{gl}_N\}\cong\mathfrak{gl}_N$, $\mathfrak{p}=\{(x,-x)\mid x\in\mathfrak{gl}_N\}$. The $\mathfrak{k}$-module $\mathfrak{p}$ is isomorphic to the $\operatorname{ad}\mathfrak{gl}_N$-module $\mathfrak{gl}_N$; thus, $G.y\cap\mathfrak{p}=K.y$ for $y=(x,-x)\in\mathfrak{p}$. Suppose that $y=(x,-x)$ is nilpotent, i.e. $x\in\mathfrak{gl}_N$ is nilpotent. The elements x and $-x$ share the same Young diagram $\boldsymbol{\mu}=(\mu_1\geqslant\cdots\geqslant\mu_k)$ and the orbit $K.y$ is uniquely determined by $\boldsymbol{\mu}$.

3.1.2 Case AI

Let χ be the nondegenerate symmetric bilinear form on V defined, in the basis $\mathbf{v}$, by:

$$\chi(v_j^{(i)},v_k^{(l)})=\begin{cases}1 & \text{if } l=i \text{ and } j+k=\lambda_i+1;\\ 0 & \text{otherwise.}\end{cases}$$

Set

$$\mathfrak{k}=\{k\in\mathfrak{g}\mid \forall u,v\in V,\ \chi(k.u,v)=-\chi(u,k.v)\}\cong\mathfrak{so}_N,$$

$$\mathfrak{p}=\{p\in\mathfrak{g}\mid \forall u,v\in V,\ \chi(p.u,v)=\chi(u,p.v)\}.$$

The symmetric Lie algebra $(\mathfrak{g},\mathfrak{k},\mathfrak{p})$ is of type AI with associated involution θ on $\mathfrak{g}$ having $\mathfrak{k}$ (resp. $\mathfrak{p}$) as $+1$ (resp. -1) eigenspace. In particular $\mathfrak{z}(\mathfrak{g})=\Bbbk\mathrm{Id}\subset\mathfrak{p}$.
In this case, each $(\mathfrak{q}_i,\mathfrak{k}\cap\mathfrak{q}_i)$ is a simple symmetric pair of type AI isomorphic to $(\mathfrak{gl}_{\lambda_i},\mathfrak{so}_{\lambda_i})$. Denote by s_k the $(k\times k)$-matrix with entries equal to 1 on its antidiagonal and 0 elsewhere, as in [GW, 3.2]. The involution θ associated to $(\mathfrak{g},\mathfrak{k},\mathfrak{p})$ acts on each element $x\in\mathfrak{q}_i$ by $\theta(x)=-s_{\lambda_i}{}^t x s_{\lambda_i}$ (which is the opposite of the symmetric matrix of x with respect to the antidiagonal). The group $\tilde{G^\theta}=\rho^{-1}(G^\theta)$, cf. (3.1), is a nonconnected group isomorphic to the orthogonal group O_N and $G^\theta\cong\mathrm{O}_N/\{\pm\mathrm{Id}\}$. Fix $\tilde\omega\in\tilde{G^\theta}\smallsetminus(\tilde{G^\theta})^\circ$, then: $\tilde{G^\theta}=(\tilde{G^\theta})^\circ\sqcup\tilde\omega(\tilde{G^\theta})^\circ$, $G^\theta=K\cup\omega K$, where $\omega=\rho(\tilde\omega)$. When N is odd, $\omega\in K=G^\theta\cong\mathrm{SO}_N$ and G^θ is connected. If N is even, one has $G^\theta=K\sqcup\omega K$ and $K\cong\mathrm{SO}_N/\{\pm\mathrm{Id}\}$.

Let $(\mathfrak{g},\mathfrak{k}',\mathfrak{p}')$ be another symmetric Lie algebra of type AI, then $\mathcal{O}\cap\mathfrak{p}'\neq\emptyset$ and, moreover, for any element $e'\in\mathcal{O}\cap\mathfrak{p}'$ there exists an isomorphism of symmetric Lie algebras $\tau:(\mathfrak{g},\mathfrak{k}',\mathfrak{p}')\to(\mathfrak{g},\mathfrak{k},\mathfrak{p})$ such that $\tau(e')=e$ (see [GW]).

3.1.3 Case AII

Assume that θ' is an involution of type AII on $\mathfrak{g}$ such that $\theta'(e) = -e$; the following condition is then necessarily satisfied:

$$\lambda_{2i+1} = \lambda_{2i+2} \text{ for all } i.$$

We therefore assume, in this subsection, that the previous condition holds. In particular, N is even and we write $N = 2N'$.
Define a symplectic form χ on V by setting

$$\chi(v_j^{(i)}, v_k^{(l)}) = \begin{cases} 1 & \text{if } i+1 = l \equiv 0 \pmod 2 \text{ and } j+k = \lambda_i + 1; \\ -1 & \text{if } l+1 = i \equiv 0 \pmod 2 \text{ and } j+k = \lambda_i + 1; \\ 0 & \text{otherwise.} \end{cases}$$

The subspaces $\mathfrak{k}$ and $\mathfrak{p}$ are then defined, through χ, as in the AI case and, θ being the associated involution, one has:

$$\mathfrak{k} \cong \mathfrak{sp}_{2N'}, \quad K = G^\theta \overset{\rho}{\cong} \tilde{G}^\theta \cong \mathrm{Sp}_{2N'}, \quad \mathfrak{z}(\mathfrak{g}) \subset \mathfrak{p}.$$

Set $\mathfrak{q}'_{2i+1} = \mathfrak{gl}(v_j^{(2i+1)}, v_j^{(2i+2)} \mid j = 1, \dots, \lambda_{2i+1})$; then, $(\mathfrak{q}'_{2i+1}, \mathfrak{k} \cap \mathfrak{q}'_{2i+1})$ is a simple symmetric pair of type AII isomorphic to $(\mathfrak{gl}_{2\lambda_{2i+1}}, \mathfrak{sp}_{2\lambda_{2i+1}})$. We can identify $\mathfrak{q}_{2i+1}$ with $\mathfrak{q}_{2i+2}$ via the isomorphism $u_i : \mathfrak{q}_{2i+1} \xrightarrow{\sim} \mathfrak{q}_{2i+2}$ defined as follows:

$$u_i(x).v_j^{(2i+2)} = x.v_j^{(2i+1)} \text{ for all } j \in [\![1, \lambda_{2i+1}]\!] \text{ and } x \in \mathfrak{q}_{2i+1}.$$

The involution θ associated to $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ acts on each element $x \in \mathfrak{q}_{2i+2}$, resp. $x \in \mathfrak{q}_{2i+1}$, by $\theta(x) = -u_i^{-1}(s_{\lambda_{2i+2}} {}^t x s_{\lambda_{2i+2}})$, resp. $\theta(x) = -u_i(s_{\lambda_{2i+1}} {}^t x s_{\lambda_{2i+1}})$.

As in case AI, if $(\mathfrak{g}, \mathfrak{k}', \mathfrak{p}')$ is another symmetric pair of type AII then $\mathcal{O} \cap \mathfrak{p}' \neq \emptyset$ and, for any element $e' \in \mathcal{O} \cap \mathfrak{p}'$, there exists an isomorphism of symmetric pairs $\tau : (\mathfrak{g}, \mathfrak{k}', \mathfrak{p}') \to (\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ with $\tau(e') = e$.

3.1.4 Case AIII

Following [Oh1, Oh2] we will use the notion of *ab*-diagram to classify nilpotent orbits in classical reductive symmetric pairs of type AIII, i.e. $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}_N, \mathfrak{gl}_p \times \mathfrak{gl}_q)$.

Definition 3.1.1. An *ab*-diagram is a Young diagram in which each box is labeled by an a or a b, in such a way that these two symbols alternate along rows. Two *ab*-diagrams are considered to be equivalent if they differ by permutations of lines of the same length.

Recall that $\mathcal{O} \subset \mathfrak{g}$ is a nilpotent orbit with associated partition $\boldsymbol{\lambda}$. To any function

$$\Phi : [\![1, \delta_{\mathcal{O}}]\!] \longrightarrow \{a, b\}$$

one can associate an *ab*-diagram $\Delta(\Phi)$ of shape $\boldsymbol{\lambda}$ as follows: label the first box of the i-th row (of size λ_i) of $\Delta(\Phi)$ by $\Phi(i)$, and continue the labeling to get an *ab*-diagram as defined above.

Observe that we may have $\Phi \neq \Psi$ and $\Delta(\Phi) = \Delta(\Psi)$.
Fix a such a function Φ and decompose V in a direct sum $V = V_a^\Phi \oplus V_b^\Phi$ by defining (cf. [Oh2])

$$\begin{aligned} V_a^\Phi &= \langle v_j^{(i)} \mid (\Phi(i) = a \text{ and } \lambda_i - j \equiv 0 \bmod 2) \text{ or } (\Phi(i) = b \text{ and } \lambda_i - j \equiv 1 \bmod 2)\rangle \\ V_b^\Phi &= \langle v_j^{(i)} \mid (\Phi(i) = b \text{ and } \lambda_i - j \equiv 0 \bmod 2) \text{ or } (\Phi(i) = a \text{ and } \lambda_i - j \equiv 1 \bmod 2)\rangle. \end{aligned}$$

Set $N_a = \dim V_a^\Phi$ and $N_b = \dim V_b^\Phi$, hence $N = N_a + N_b$. Now, if

$$\mathfrak{k}^\Phi = \mathfrak{gl}(V_a^\Phi) \oplus \mathfrak{gl}(V_b^\Phi) \subset \mathfrak{g}, \quad \mathfrak{p}^\Phi = \mathrm{Hom}(V_a^\Phi, V_b^\Phi) \oplus \mathrm{Hom}(V_b^\Phi, V_a^\Phi) \subset \mathfrak{g}$$

we obtain a symmetric Lie algebra

$$(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}) = (\mathfrak{g}, \mathfrak{k}^\Phi, \mathfrak{p}^\Phi),$$

such that $([\mathfrak{g},\mathfrak{g}], \mathfrak{k} \cap [\mathfrak{g},\mathfrak{g}])$ is irreducible of type AIII and $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{k}$. One has: $K = \rho(\mathrm{GL}(V_a^\Phi) \times \mathrm{GL}(V_b^\Phi))$ and, θ being the associated involution, $K = G^\theta$ if and only if $N_a \neq N_b$. It is easily seen that $(\mathfrak{q}_i, \mathfrak{k}^\Phi \cap \mathfrak{q}_i)$ is a reductive symmetric pair (of type AIII) isomorphic to $(\mathfrak{gl}_{\lambda_i}, \mathfrak{gl}_{\lfloor \frac{\lambda_i}{2} \rfloor} \oplus \mathfrak{gl}_{\lceil \frac{\lambda_i}{2} \rceil})$.

The ab-diagram associated to a nilpotent element $e' \in \mathfrak{p}^\Phi$ is defined in the following way (see, for example, [Oh2, (1.4)]). Let $\boldsymbol{\mu} = (\mu_1 \geqslant \cdots \geqslant \mu_k)$ be the partition associated to e'. Fix a normal $\mathfrak{sl}_2$-triple (e', h', f') and a basis of V

$$\{\zeta_j^{(i)} \mid i \in [\![1, k]\!],\ j \in [\![1, \mu_i]\!]\}$$

such that: $\zeta_j^{(i)}$ belongs either to V_a^Φ or V_b^Φ, $(\zeta_1^{(i)})_i$ is a basis of $\ker f'$ and $e'(\zeta_j^{(i)}) = \zeta_{j+1}^{(i)}$. Then, label the j-th box in the i-th row of the Young diagram associated to $\boldsymbol{\mu}$ by a, resp. b, if $\zeta_j^{(i)} \in V_a^\Phi$, resp. $\zeta_j^{(i)} \in V_b^\Phi$. This ab-diagram is uniquely determined by e' and will be denoted by $\Gamma^\Phi(e')$. The map $K.x \mapsto \Gamma^\Phi(x)$ gives a parameterization of the nilpotent K-orbits in $\mathfrak{p}^\Phi$, see [Oh2, Proposition 1(2)].
Remark that the element e_i, defined in 1.6.1, belongs to $\mathfrak{p}^\Phi \cap \mathfrak{q}_i$ in the symmetric Lie algebra $(\mathfrak{q}_i, \mathfrak{k}^\Phi \cap \mathfrak{q}_i)$; its ab-diagram has only one row, with first box labeled with $\Phi(i)$. An ab-diagram of the form $\Gamma^\Phi(x)$ is said to be *admissible* for Φ. For example, $\Gamma^\Phi(e) = \Delta(\Phi)$ is admissible. It is easy to see that a necessary and sufficient condition for an ab-diagram to be admissible is to have exactly N_a labels equal to a and N_b labels equal to b.
The number $N_a - N_b$ is called the *parameter* of the symmetric pair $(\mathfrak{g}, \mathfrak{k}^\Phi)$. Its absolute value $|N_a - N_b|$ can be read from the Satake diagram of the symmetric pair $(\mathfrak{g}, \mathfrak{k}^\Phi)$. The parameter is different from 0 when all the white nodes are connected by arrows; then, its absolute value is the number of black nodes plus one, cf. [He1, p. 532]. Two symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ of type AIII are isomorphic if and only if their parameters have the same absolute value.

Assume that $(\mathfrak{g}, \mathfrak{k}', \mathfrak{p}')$ is a symmetric Lie algebra of type AIII such that $\mathcal{O} \cap \mathfrak{p}' \neq \emptyset$. Then, for every element $e' \in \mathcal{O} \cap \mathfrak{p}$ with ab-diagram Γ', there exists a function $\Psi : [\![1, \delta_{\mathcal{O}}]\!] \to \{a, b\}$ such that Γ' is admissible for Ψ. Furthermore, it is not difficult to show that, in this case, there exists an isomorphism of symmetric Lie algebras $\tau : (\mathfrak{g}, \mathfrak{k}', \mathfrak{p}') \to (\mathfrak{g}, \mathfrak{k}^\Psi, \mathfrak{p}^\Psi)$ such that $\tau(e') = e$.

3.1.5 Notation and remarks

Let $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ be a symmetric Lie algebra with $\mathfrak{g} = \mathfrak{gl}_N = \mathfrak{gl}(V)$ and S_G be a G-sheet intersecting $\mathfrak{p}$. We follow the notation introduced in sections 1.6.1 and 2.5.
Recall that the nilpotent orbit $\mathcal{O} \subset S_G$ intersects $\mathfrak{p}$ and fix $e \in \mathcal{O} \cap \mathfrak{p}$. Then, the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ can be described as in 3.1.2, 3.1.3 or 3.1.4. The notation for $\mathbf{v}$, $\mathfrak{q} = \bigoplus_i \mathfrak{q}_i$, $\mathfrak{l}$, $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{l} \cap \mathfrak{q}$, being as in 1.6.1, set:

$$\mathfrak{k}_i = \mathfrak{q}_i \cap \mathfrak{k}, \quad \mathfrak{p}_i = \mathfrak{q}_i \cap \mathfrak{p}, \quad \mathfrak{p}_i = \mathfrak{q}_i \cap \mathfrak{p}, \quad \theta_i = \theta_{|\mathfrak{q}_i}.$$

The normal $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, f, h)$ is then given by $e = \sum_i e_i$, $h = \sum_i h_i$, $f = \sum_i f_i$. The map

$$\varepsilon = \varepsilon^{\mathfrak{g}} : e + \mathfrak{h} \to e + \mathfrak{g}^f$$

is defined as in Remark 1.4.5; it is the restriction of the polynomial map ϵ from Lemma 1.4.4.

Recall also that the subset $\mathsf{Z} \subset G$ is chosen such that: $\mathrm{Id} \in \mathsf{Z}$, $\{g.e\}_{g \in \mathsf{Z}}$ is a set of representatives of the K-orbits contained in $G.e \cap \mathfrak{p}$ and $g.\mathscr{S} = (g.e, g.h, g.f)$ is a normal $\mathfrak{sl}_2$-triple. The "Slodowy slices" are defined by:

$$g.e + X(S_G, g.\mathscr{S}) = S_G \cap (g.e + \mathfrak{g}^{g.f}), \quad X_{\mathfrak{p}}(S_G, g.\mathscr{S}) = X(S_G, g.\mathscr{S}) \cap \mathfrak{p}.$$

As observed in Remark 1.4.3, we may simplify the notation by setting:

$$X = X(\mathscr{S}) = X(S_G, \mathscr{S}), \quad X_{\mathfrak{p}} = X_{\mathfrak{p}}(\mathscr{S}) = X(S_G, \mathscr{S}) \cap \mathfrak{p}.$$

It follows from the results of Section 1.4 that: X is smooth, $e + X = \varepsilon(e + \mathfrak{t})$ is irreducible, $S_G = G.(e + X)$ and $\psi : S_G \to e + X$ is a geometric quotient of the sheet S_G, cf. Theorem 1.4.7 (recall that the group G^e is connected).

Since $e, g.e \in \mathfrak{p}$, the remarks at the end of the previous subsections show that there exists an isomorphism τ (depending on g) of symmetric Lie algebras sending $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ to a symmetric pair of the same type and e to $g.e$. It is not hard to see that we can further assume that $\tau(\mathscr{S}) = g.\mathscr{S}$. The main consequence of this observation is that, applying τ, any property obtained for $e + X_{\mathfrak{p}}(\mathscr{S})$ also holds for $g.e + X_{\mathfrak{p}}(g.\mathscr{S})$. In particular, we will mainly work with $e + X_{\mathfrak{p}}(\mathscr{S})$.

3.2 Properties of slices

We continue with the notation of 3.1.5. Hence $S_G \subset \mathfrak{g}$ is a G-sheet, $e \in S_G \cap \mathfrak{p}$ is a fixed nilpotent element and $\mathscr{S} = (e, f, h)$, $\mathbf{v}$, $\mathfrak{q}$, etc., are as defined in 1.6.1.

3.2.1 The slice property (1)

In this subsection we give a proof of Theorem 3.2.1 for types AI and AII. It asserts that the condition ($\heartsuit$) holds; we also refer to it as the *slice property*.

Theorem 3.2.1. *Assume that* $(\mathfrak{g}, \theta)$ *is of type* A*. Then, one has:*

$$G.(e + X_{\mathfrak{p}}) = G.(S_G \cap \mathfrak{p}). \tag{$\heartsuit$}$$

Moreover, in types AI *and* AII *a stronger version holds, namely:* $e + X \subset \mathfrak{p}$.

Proof. Since $S_G = G.(e + X)$ and $e + X_{\mathfrak{p}} = (e + X) \cap \mathfrak{p} \subset S_G \cap \mathfrak{p}$, the inclusion $G.(e + X_{\mathfrak{p}}) \subset G.(S_G \cap \mathfrak{p})$ is obvious. Clearly, $e + X \subset \mathfrak{p}$ yields $G.(e + X_{\mathfrak{p}}) = G.(e + X) = S_G \supset G.(S_G \cap \mathfrak{p})$. We prove below that the inclusion $e + X \subset \mathfrak{p}$ is true when $(\mathfrak{g}, \mathfrak{k})$ is of type AI or AII. The proof of the theorem in type AIII is postponed to subsection 3.2.2, see Proposition 3.2.6.

Type AI: As said in subsection 3.1.2, each $(\mathfrak{q}_i, \mathfrak{k}_i)$ is a symmetric pair of type AI. Since this pair has maximal rank and $e_i \in \mathfrak{p}_i$ is a regular element, one has $\mathfrak{q}_i^{f_i} = \mathfrak{p} \cap \mathfrak{q}_i^{f_i}$. Therefore the image of each map $\varepsilon_i : e_i + \mathfrak{h}_i \to e_i + \mathfrak{q}^{f_i}$, as defined in 1.6.1, is contained in $\mathfrak{q}_i^{f_i} \subseteq \mathfrak{p}_i$. From $\varepsilon = \sum_i \varepsilon_i$ one gets that $e + X = \varepsilon(e + \mathfrak{t}) \subset S_G \cap \mathfrak{p}$.

Type AII: Recall that $\lambda_{2i+1} = \lambda_{2i+2}$ if $2i + 2 \leqslant \delta_{\mathcal{O}}$. Let $x = \sum_i x_i \in \mathfrak{q}$; then $x \in \mathfrak{p} \cap \mathfrak{q}$ if and only if, for all i, $x_{2i+1} = -\theta_{2i+2}(x_{2i+2})$, which is the symmetric of $u_i^{-1}(x_{2i+2})$ with respect to the antidiagonal (cf. §3.1.3). Fix $t \in \mathfrak{t}$, hence $e + t \in S_G$; from the description of $\mathfrak{t}$ given in (1.7), one deduces that $u_i(e_{2i+1} + t_{2i+1}) = e_{2i+2} + t_{2i+2}$. Set $x = \varepsilon(e + t)$. It follows from $u_i \circ \varepsilon_{2i+1} = \varepsilon_{2i+2} \circ u_i$ that $u_i(x_{2i+1}) = x_{2i+2}$. Since $e_{2i+1} + \mathfrak{q}_{2i+1}^{f_{2i+1}}$ is fixed under the conjugation by $s_{\lambda_{2i+1}}$, one obtains $-\theta_{2i+2}(x_{2i+2}) = s_{\lambda_{2i+1}}{}^t x_{2i+1} s_{\lambda_{2i+1}} = x_{2i+1}$. Hence $\varepsilon(e + t) \in \mathfrak{p}$ and, therefore, $\varepsilon(e + \mathfrak{t}) = e + X \subset \mathfrak{p}$. □

Corollary 3.2.2. *Every* G*-orbit contained in* S_G *and intersecting* $\mathfrak{p}$*, also intersects* $(\mathfrak{q} \cap \mathfrak{p})^\bullet$*.*

Proof. It suffices to observe that $e + X \subset \mathfrak{q}^\bullet$ and $(\mathfrak{q} \cap \mathfrak{p})^\bullet \subset \mathfrak{q}^\bullet$ □

Remark 3.2.3. (1) One can deduce Theorem 3.2.1 from Corollary 3.2.2. Indeed, let $x \in S_G$ and suppose that $y \in G.x \cap (\mathfrak{q} \cap \mathfrak{p})^\bullet$. Since e is regular in $\mathfrak{q}$, it follows from [KR] that y is $(Q \cap K)^\circ$-conjugate to an element of $e + X_{\mathfrak{p}}$.
(2) Assume that $(\mathfrak{g}, \mathfrak{k})$ is of type AI or AII. Then, since $e + X_{\mathfrak{p}} = e + X$ is irreducible and smooth in type A (see §3.1.5), Theorem 3.2.1 yields that the conditions ($\diamondsuit$) and (2.7), cf. §2.5, hold.

3.2.2 The slice property (2)

We assume in this section that $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p}) = (\mathfrak{g}, \mathfrak{k}^\Phi, \mathfrak{p}^\Phi)$ is of type AIII. Let $\mathfrak{a} \subset \mathfrak{p}$ be Cartan subspace and $\mathfrak{h}' \subset \mathfrak{g}$ be a Cartan subalgebra containing $\mathfrak{a}$. Denote by B a σ-fundamental system of the root system $R(\mathfrak{g}, \mathfrak{h}') = R([\mathfrak{g}, \mathfrak{g}], \mathfrak{h}' \cap [\mathfrak{g}, \mathfrak{g}])$, see §2.2 with $\mathfrak{h}$ replaced by $\mathfrak{h}' \cap [\mathfrak{g}, \mathfrak{g}]$. Let $\bar{D}$ be the Satake diagram of type AIII associated to B (cf. [He1, p. 532]). Since $\mathfrak{a} \subset [\mathfrak{g}, \mathfrak{g}]$, see 3.1.4, one can define a $\mathbb{Q}$-form of $\mathfrak{a}$ by

$$\mathfrak{a}_{\mathbb{Q}} = \{a \in \mathfrak{a} \mid \ \alpha(a) \in \mathbb{Q} \text{ for all } \alpha \in R(\mathfrak{g}, \mathfrak{h}')\}.$$

The nodes of $\bar{D}$ can be labeled by the elements $\alpha_1, \dots, \alpha_{N-1}$ of B. Set $\alpha'_i = \alpha_{N-i}$, $1 \le i \le N-1$, hence $\alpha_{i|\mathfrak{a}} = \alpha'_{i|\mathfrak{a}}$; there exists an arrow between α_i and α'_i when these nodes are colored in white and $i \neq N/2$.

Let $s \in \mathfrak{g}$ be semisimple and let $c \in \mathsf{sp}(s)$ be an eigenvalue of s on V. Denote by $V_{s,c}$ the eigenspace associated to c; thus, $m(s,c) = \dim V_{s,c}$ is the multiplicity of c. More generally, see §1.6.2, we set $V_{s,d} = \ker(s - d\,\mathrm{Id}_V)$ and $m(s,d) = \dim V_{s,d}$ for every $d \in \Bbbk$. One can identify $\mathfrak{gl}(V_{s,c})$ with a Lie subalgebra of $\mathfrak{gl}(V)$ by extending an element $x \in \mathfrak{gl}(V_{s,c})$ by 0 on $\bigoplus_{c' \neq c} V_{s,c'}$. Under this identification, $\mathfrak{sl}(V_{s,c})$ is a simple factor of $\mathfrak{l}$ if and only if $m(s,c) \geqslant 2$. Setting

$$\mathfrak{w}'_{s,c} = \mathfrak{sl}(V_{s,c}), \quad \mathfrak{w}_{s,c} = \mathfrak{gl}(V_{s,c}),$$

one has:

$$\mathfrak{g}^s = \bigoplus_{c \in \mathsf{sp}(s)} \mathfrak{w}_{s,c} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s) \oplus \bigoplus_{m(s,c) \geqslant 2} \mathfrak{w}'_{s,c}. \tag{3.2}$$

Denote by $M_{s,c}$ the connected algebraic subgroup of G group with Lie algebra $\mathfrak{w}'_{s,c}$. Then, $M_{s,c}$ acts on $\mathfrak{w}_{s,c}$ via the adjoint action and the group G^s is generated by $C_G(\mathfrak{g}^s)$ and the $M_{s,c}$, $c \in \mathsf{sp}(s)$ (see §1.2 and Proposition 1.2.3).

The group $\{\pm 1\}$ acts by multiplication on $\mathsf{sp}'(s) = \{c \in \mathsf{sp}(s) \mid -c \in \mathsf{sp}(s)\}$; let $\mathsf{sp}_{\pm}(s) = \mathsf{sp}'(s)/\{\pm 1\}$ be the orbit space. The class of $c \in \mathsf{sp}'(s)$ in $\mathsf{sp}_{\pm}(s)$ is denoted by $\pm c$. When $0 \in \mathsf{sp}(s)$ we simply write $\pm 0 = 0$. We then set

$$\mathfrak{g}_{s,\pm c} = \mathfrak{w}_{s,c} \oplus \mathfrak{w}_{s,-c}, \quad \mathfrak{g}_{s,0} = \mathfrak{w}_{s,0}.$$

If $0 \neq c \in \mathsf{sp}'(s)$, the connected subgroup of G generated by $M_{s,c}$ and $M_{s,-c}$ is denoted by $G_{s,\pm c}$ and we set $G_{s,0} = M_{s,0}$. One has $\mathrm{Lie}(G_{s,\pm c}) = [\mathfrak{g}_{s,\pm c}, \mathfrak{g}_{s,\pm c}]$.

Recall that we have written $V = V_a^{\Phi} \oplus V_b^{\Phi}$; we set $V_a = V_a^{\Phi}$, $V_b = V_b^{\Phi}$. The parameter of $(\mathfrak{g}, \mathfrak{k})$ is $N_a - N_b$ where $N_a = \dim V_a$, $N_b = \dim V_b$, see 3.1.4.

Lemma 3.2.4. *Let $s \in \mathfrak{p}$ be a semisimple element. Then:*
(1) *$m(s,c) = m(s,-c)$ for all $c \in \Bbbk$;*
(2) *the symmetric Lie algebra $(\mathfrak{g}^s, \mathfrak{k}^s)$ decomposes as $\bigoplus_{\pm c \in \mathsf{sp}_{\pm}(s)} (\mathfrak{g}_{s,\pm c}, \mathfrak{k}_{s,\pm c})$, where $\mathfrak{k}_{s,\pm c} = \mathfrak{k} \cap \mathfrak{g}_{s,\pm c}$;*
(3) *if $c \neq 0$, $(\mathfrak{g}_{s,\pm c}, \mathfrak{k}_{s,\pm c})$ is a reductive symmetric pair whose semisimple part is irreducible of type* A0*;*
(4) *$V_{s,0} = (V_{s,0} \cap V_a) \oplus (V_{s,0} \cap V_b)$ and the symmetric Lie algebra $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$ is a reductive symmetric pair whose semisimple part is irreducible of type* AIII*, with the same parameter as $(\mathfrak{g}, \mathfrak{k})$. In particular, the parameter of $(\mathfrak{g}, \mathfrak{k})$ is 0 when $0 \notin \mathsf{sp}(s)$.*

Proof. (1) Since the involution θ is inner, the claim follows from the following elementary observation. Suppose that $A \in \mathrm{GL}_N$, $x \in \mathfrak{gl}_N$, and set $x' = AxA^{-1}$. Then, $m(x,c) = \dim\ker(x - c\,\mathrm{Id}) = \dim\ker(x' - c\,\mathrm{Id})$; in particular, $m(x,c) = m(x',c)$, thus $m(x,c) = m(x,-c)$ when $x' = -x$.

(2) The assertion is an easy consequence of (3.2) and $\theta(\mathfrak{w}_{s,c}) = \mathfrak{w}_{s,-c}$.
(3) & (4). We may assume that $N_a \geqslant N_b$ and, by Proposition 2.3.5, $s \in \mathfrak{a}_{\mathbb{Q}}$. Then, the claims can be read on the Satake diagram of type AIII, except for the equality of the parameters when $c = 0$ (one only sees in this way that the absolute values are equal). A complete proof can be given as follows.
Let $(v_{a,i})_{i\in[\![1,N_a]\!]}$ and $(v_{b,i})_{i\in[\![1,N_b]\!]}$ be bases of V_a and V_b. For each $i \in [\![1, N_b]\!]$, define $u_i \in \mathfrak{p}$ by

$$u_i(v_{d,j}) = \begin{cases} v_{\bar{d},i} & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{d}$ is the element of $\{a, b\} \smallsetminus \{d\}$. The subspace generated by the u_i, $i \in [\![1, N_b]\!]$, is a Cartan subspace of $\mathfrak{p}$. If $s = \sum_i c_i u_i$, the eigenvalues of s are given by square roots of the c_i's and one has $V_{s,0} = \langle \{v_{a,i}, v_{b,i} \mid c_i = 0\} \cup \{v_{a,i} \mid i > N_b\}\rangle$. It is then not difficult to get the desired assertions. □

Recall from §1.6.2 that if $t = \sum_i t_i \in \mathfrak{q} = \bigoplus_i \mathfrak{q}_i$ is semisimple, $m_i(t, c)$ denotes the multiplicity of the eigenvalue c for $t_i \in \mathfrak{q}_i$; recall also that $\mathfrak{h} \subset \mathfrak{q}$.

Lemma 3.2.5. *Let $t \in \mathfrak{h}$ be such that $G.(e + t) \cap \mathfrak{p} \neq \emptyset$. Then:*

$$m_i(t, c) = m_i(t, -c) \quad \textit{for all } c \in \Bbbk. \tag{3.3}$$

Proof. Let $s_1 + n_1$ be the Jordan decomposition of $e + t$ and pick $g \in G$ such that $g.(e + t) \in \mathfrak{p}$. Therefore, $s = g.s_1 \in \mathfrak{p}$ and $n = g.n_1 \in \mathfrak{p} \cap \mathfrak{g}^s$. By Corollary 1.6.4 we know that t, s_1 and s are in the same G-orbit. Then, Lemma 3.2.4(1) gives $m(t, c) = m(s, c) = m(s, -c) = m(t, -c)$. On the other hand, $n \in \mathfrak{p} \cap \mathfrak{g}^s$ is a nilpotent element of the subsymmetric pair $(\mathfrak{g}^s, \mathfrak{g}^s \cap \mathfrak{k}) = \prod_{\pm c\in \mathsf{sp}_\pm(s)}(\mathfrak{g}_{s,\pm c}, \mathfrak{k}_{s,\pm c})$, cf. Lemma 3.2.4(3,4). With obvious notation, one can decompose the orbit $K.n$ of this direct product as follows:

$$K.n = \prod_{\pm c\in \mathsf{sp}_\pm(s)} \mathcal{O}_{\pm c}.$$

The result in the case $c = 0$ is clear. Recall that when $c \neq 0$ one has $\mathfrak{g}_{s,\pm c} = \mathfrak{w}_{s,c} \oplus \mathfrak{w}_{s,-c}$, and we can further decompose each orbit $G_{s,\pm c} \cdot \mathcal{O}_{\pm c}$ as $\mathcal{O}_c \times \mathcal{O}_{-c} \subset \mathfrak{w}_{s,c} \times \mathfrak{w}_{s,-c}$. Then, $G_{s,\pm c} \cdot \mathcal{O}_{\pm c}$, is characterized by the Young diagrams of the nilpotent orbits $\mathcal{O}_c$, $\mathcal{O}_{-c}$. Since $(\mathfrak{g}_{s,\pm c}, \mathfrak{k}_{s,\pm c})$ is of type A0, these two Young diagrams are equal (cf. §3.1.1). The results of §1.6.4 then yield that the partition of $\mathcal{O}_{\delta c}$, $\delta \in \{-1, 1\}$, is given by the sequence $(m_i(t, \delta c))_i$. As these two sequences are decreasing on i, cf. (1.7), one obtains $m_i(t, c) = m_i(t, -c)$ for all i. □

The following proposition completes the proof of Theorem 3.2.1 and Corollary 3.2.2 in case AIII.

Proposition 3.2.6. *Let $e+t \in e+\mathfrak{t}$.*
(i) *If t satisfies (3.3), then $\varepsilon(e+t) \in e+X_{\mathfrak{p}}$.*
(ii) *One has $G.(e+t) \cap \mathfrak{p} \neq \emptyset$ if and only if t satisfies (3.3).*
(iii) *The condition ($\heartsuit$) holds, i.e. $G.(S_G \cap \mathfrak{p}) = G.(e+X_{\mathfrak{p}})$.*

Proof. (i) Recall that $t = \sum_i t_i$, $e = \sum_i e_i$ with $t_i \in \mathfrak{q}_i$ and $e_i \in \mathfrak{p} \cap \mathfrak{q}_i$ regular in $\mathfrak{q}_i \cong \mathfrak{gl}_{\lambda_i}$. The map ε can be written as $\sum_i \varepsilon_i$, where ε_i is given by Lemma 1.6.2 applied in the algebra $\mathfrak{q}_i$. Thus $\varepsilon_i(e_i+t_i) = e_i + \sum_{j \leq 0} P_j(t_i)$. From (3.3) and the symmetry of the polynomials P_j one obtains $P_j(t_i) = 0$ if j is even. One can deduce from the construction made in 3.1.4 that the subspaces $\mathfrak{p}_i = \mathfrak{p} \cap \mathfrak{q}_i$ are the sum of the j-subdiagonals and j-supdiagonals of $\mathfrak{q}_i$ for j odd. It follows that $\varepsilon_i(e_i+t_i) \in e_i + \mathfrak{p}_i^{f_i}$, hence $\varepsilon(e+t) \in e + X \cap \mathfrak{p}$.
(ii) By Lemma 1.4.4 one has $G.(e+t) = G.\varepsilon(e+t)$, thus part (i) shows that the condition is sufficient. Lemma 3.2.5 gives the converse.
(iii) The inclusion $G.(e+X_{\mathfrak{p}}) \subset G.(S_G \cap \mathfrak{p})$ is always true. By Proposition 1.4.2, every $x \in S_G \cap \mathfrak{p}$ is G-conjugate to an element $e+t \in e+\mathfrak{t}$; parts (i) and (ii) give $\varepsilon(e+t) \in G.x \cap (e+X_{\mathfrak{p}})$ and the result follows. $\square$

We now find a convenient subspace $\mathfrak{c} \subset \mathfrak{t}$ such that $\varepsilon(e+\mathfrak{c}) = e+X_{\mathfrak{p}}$. For $i \in [\![1, \lambda_{\delta_{\mathcal{O}}}]\!]$ and $j \in [\![0, \lfloor(\lambda_i - \lambda_{i+1})/2\rfloor - 1]\!]$, define elements $c(i,j) = (c(i,j)_k)_k \in \Bbbk^{\lambda_1}$ by:

$$c(i,j)_k = \begin{cases} 1 & \text{if } k = \lambda_{i+1} + 2j + 1; \\ -1 & \text{if } k = \lambda_{i+1} + 2j + 2; \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Let $\mathfrak{c}'$ be the subspace of $\Bbbk^{\lambda_1}$ generated by the elements $c(i,j)$. Recall from (1.8) the isomorphism $\alpha : \Bbbk^{\lambda_1} \xrightarrow{\sim} \mathfrak{t}$ and set:

$$\mathfrak{c} = \alpha(\mathfrak{c}') \subset \mathfrak{t}. \tag{3.5}$$

The main property of the subspace $\mathfrak{c}$ is the following. By construction every element of $\mathfrak{c}$ satisfies (3.3); conversely, Lemma 1.5.1 applied in each $\mathfrak{q}_i$ implies that any element $e+t$ (with $e = \sum_i e_i$, $t = \sum_i t_i$) satisfying (3.3) is conjugate to an element of $\mathfrak{c}$.

Proposition 3.2.7. *Under the previous notation one has: $\varepsilon(e+\mathfrak{c}) = e+X_{\mathfrak{p}}$ and $G.(e+\mathfrak{c}) = G.(S_G \cap \mathfrak{p})$. Moreover,*

$$\dim \mathfrak{c} = \sum_{i=1}^{\delta_{\mathcal{O}}} \left\lfloor \tfrac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor, \tag{3.6}$$

which only depends on $\boldsymbol{\lambda}$.

Proof. The formula (3.6) follows without difficulty from the definition of $\mathfrak{c}'$. Since the elements of $e+\mathfrak{c}$ satisfy (3.3), Proposition 3.2.6(i) gives $\varepsilon(e+\mathfrak{c}) \subset e+X_{\mathfrak{p}}$. Conversely, let $e+x \in e+X_{\mathfrak{p}}$. As $e+X = \varepsilon(e+\mathfrak{t})$, the element $e+x = \varepsilon(e+t)$, $t \in \mathfrak{t}$, is the unique point of $e+X$ intersecting the orbit $G.(e+x) = G.\varepsilon(e+t) = G.(e+t)$ (see Lemma 1.4.4(i)). By Proposition 3.2.6(ii), $e+t$ satisfies (3.3) and, as noticed above, $e+t$ is conjugate to an element $e+c \in e+\mathfrak{c} \subset e+\mathfrak{t}$.

It follows that $\{e + x\} = G.(e + x) \cap (e + X) = G.\varepsilon(e + c) \cap (e + X) = \{\varepsilon(e + c)\}$. Hence, $e + x = \varepsilon(e + c) \in \varepsilon(e + \mathfrak{c})$. Finally, $G.(S_G \cap \mathfrak{p}) = G.(e + X_{\mathfrak{p}}) = G.\varepsilon(e + \mathfrak{c}) = G.(e + \mathfrak{c})$. □

Remark 3.2.8. Proposition 3.2.7 implies that condition (♢) holds in case AIII, i.e., $e + X_{\mathfrak{p}}$ is irreducible. Actually, using similar arguments to [IH, Chapter 5] it is possible to show that $e + X_{\mathfrak{p}}$ is isomorphic to the quotient of $\mathfrak{c}$ by a reflection group of type B, hence is smooth. In particular, since G^e is connected, the stronger condition (2.7) holds.

Corollary 3.2.2 says that in each G-orbit contained in S_G and intersecting $\mathfrak{p}$ one can find an element $x = s + n \in (\mathfrak{q} \cap \mathfrak{p})^{\bullet}$. The next corollary summarizes various results which can be deduced from Lemma 3.2.4. Recall that $\mathfrak{q} = \bigoplus_i \mathfrak{q}_i$ and that $(\mathfrak{q}_i, \mathfrak{k} \cap \mathfrak{q}_i)$ is a symmetric Lie algebra of type AIII. Applying Lemma 3.2.4 in each symmetric pair $(\mathfrak{q}_i, \mathfrak{k} \cap \mathfrak{q}_i)$ yields:

Corollary 3.2.9. *Let $x = s + n \in (\mathfrak{q} \cap \mathfrak{p})^{\bullet}$ and write $s = \sum_i s_i$, $n = \sum_i n_i$ with $s_i, n_i \in \mathfrak{p} \cap \mathfrak{q}_i$, as in 1.6.1.*
(1) *The Levi factor $\mathfrak{q}_i^{s_i}$ of $\mathfrak{q}_i$ has the following decomposition:*

$$\mathfrak{q}_i^{s_i} = \bigoplus_{c \in \Bbbk} \mathfrak{w}_{i,s_i,c}$$

where $\mathfrak{w}_{i,s_i,c}$ is identified with $\mathfrak{gl}(\ker(s_i - c\,\mathrm{Id})) \subset \mathfrak{q}_i$.
(2) *The symmetric pair $(\mathfrak{q}_i^{s_i}, \mathfrak{q}_i^{s_i} \cap \mathfrak{k})$ decomposes as*

$$(\mathfrak{q}_i^{s_i}, \mathfrak{q}_i^{s_i} \cap \mathfrak{k}) = \bigoplus_{\pm c \in \mathsf{sp}_{\pm}(s_i)} (\mathfrak{q}_{i,s_i,\pm c}, \mathfrak{k}_{i,s_i,\pm c})$$

where $(\mathfrak{q}_{i,s_i,0}, \mathfrak{k}_{i,s_i,0}) = (\mathfrak{w}_{i,s_i,0}, \mathfrak{w}_{i,s_i,0} \cap \mathfrak{k})$ is of type AIII *and, when $c \neq 0$, $(\mathfrak{q}_{i,s_i,\pm c}, \mathfrak{k}_{i,s_i,\pm c}) = ((\mathfrak{w}_{i,s_i,c} \oplus \mathfrak{w}_{i,s_i,-c}), (\mathfrak{w}_{i,s_i,c} \oplus \mathfrak{w}_{i,s_i,-c}) \cap \mathfrak{k})$ is of type* A0.
(3) *The factor $(\mathfrak{q}_{i,s_i,0}, \mathfrak{k}_{i,s_i,0})$ has the same parameter as $(\mathfrak{q}_i, \mathfrak{q}_i \cap \mathfrak{k})$. In particular, the ranks of $\mathfrak{q}_i$ and $\mathfrak{q}_{i,s_i,0}$ have the same parity.*
(4) *The nilpotent element n_i is regular in $\mathfrak{q}_i^{s_i}$; thus, the orbit $(Q \cap K)^{\circ}.n_i$ is uniquely determined by its one row ab-diagram (see 3.1.4).*

3.3 J_K-classes in type A

Knowing that (♡) holds, we want to prove below that condition (♣), introduced in §2.5, is satisfied. As above, $S_G \subset \mathfrak{g}^{(2m)}$ is a G-sheet and $e \in S_G$ is a nilpotent element. We fix a Jordan G-class $J \subset S_G$ such that $J \cap \mathfrak{p} \neq \emptyset$. Recall from Theorem 2.4.4 that $J \cap \mathfrak{p}$ is a (disjoint) union of J_K-classes.

3.3.1 Cases AI and AII

In this subsection we assume that $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ is a symmetric Lie algebra of type AI or AII, as described in 3.1.2 and 3.1.3.

We will need the following result, which is a formulation of [Oh1, Proposition 4] in a slightly more general setting. (Its proof is exactly the same.)

Proposition 3.3.1 (Ohta). *Let κ be a linear involution of the associative algebra $\mathfrak{g} = \mathfrak{gl}_N$ and $x \mapsto x^*$ be a linear anti-involution of the associative algebra $\mathfrak{g}$ which commutes with κ. Define:*

$$G' = \mathfrak{g}^\kappa \cap \mathrm{GL}_N, \quad G'' = \{g \in G' : g^* = g^{-1}\}.$$

Set $\sigma(x) = -x^*$ and let η, η' be elements of $\{\pm 1\}$. Then, via the adjoint action, G' acts on $\mathfrak{g}^{\eta'\kappa}$ and G'' acts on $\mathfrak{g}^{\eta\sigma} \cap \mathfrak{g}^{\eta'\kappa}$. The elements $x, y \in \mathfrak{g}^{\eta\sigma} \cap \mathfrak{g}^{\eta'\kappa}$ are conjugate under G'' if and only if they are conjugate under G'.

We may apply this proposition in the two following situations. Fixing $\eta = -1$, $\eta' = 1$, we take: ($\kappa = \mathrm{Id}, x^* = {}^t x$) in type AI, ($\kappa = \mathrm{Id}, x^* = -J^t x J$) in type AII, where $J = \left[\begin{smallmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{smallmatrix}\right] \in \mathfrak{gl}_{2N'}$. Observe that $\mathfrak{g}' = \mathfrak{g} = \mathfrak{gl}_N$, $G'' = \mathrm{O}_N$, resp. $G'' = \mathrm{Sp}_N$, and that the action of $G' = \mathrm{GL}_N = \tilde{G}$ factorizes through $G \cong \tilde{G}/\{\Bbbk^\times \mathrm{Id}\}$. Then, σ is an involution of the Lie algebra $\mathfrak{g}$ of type AI, resp. AII (cf. [GW, Theorem 3.4]). Using an isomorphism τ as explained in 3.1, we may assume that $\mathfrak{k} = \tau(\mathfrak{g}^\sigma)$ and $\mathfrak{p} = \tau(\mathfrak{g}^{-\sigma})$. Moreover, in each case $\rho(\tau(G'')) = G^\theta$ (cf. 3.1.2 and 3.1.3).
We therefore have obtained the (well known) result:

Proposition 3.3.2. *Let $(\mathfrak{g}, \theta)$ be of type* AI *or* AII*. If $x, y \in \mathfrak{p}$ one has the equivalence:*

$$G^\theta.x = G^\theta.y \iff G.x = G.y.$$

Corollary 3.3.3. *If $(\mathfrak{g}, \theta)$ is of type* AI *or* AII*, the J_K-classes contained in $J \cap \mathfrak{p}$ are conjugate under G^θ.*

Proof. Let $J_1 = K.(\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet + n)$ be the Jordan K-class containing $x = s + n \in J \cap \mathfrak{p}$ and denote by $J_2 = K.(\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet + n')$ another Jordan K-class contained in $J \cap \mathfrak{p}$ (cf. 2.4.2(i)). Since $J = G.(\mathfrak{c}_\mathfrak{g}(\mathfrak{g}^s)^\bullet + n')$, there exists $g \in G$ such that $g.x \in \mathfrak{c}_\mathfrak{g}(\mathfrak{g}^s)^\bullet + n'$ and Lemma 2.3.7(ii) implies that $g.x \in J_2$. Now, by Proposition 3.3.2, we may assume that $g \in G^\theta$. Then, $g.J_1$ is an irreducible subvariety of $J \cap \mathfrak{p}$ of dimension $\dim J \cap \mathfrak{p}$ (see Lemma 2.4.2(ii)) which intersects J_2. It follows from Theorem 2.4.4 that $g.J_1 = J_2$. □

Remark 3.3.4. As $G^\theta = K \cup \omega K$ in type AI (cf. 3.1.2), there are at most two Jordan K-classes in $J \cap \mathfrak{p}$. In type AII one has $G^\theta = K$ and $J \cap \mathfrak{p}$ is a Jordan K-class.

Corollary 3.3.5. *The condition (♣) of section 2.5 is satisfied.*

Proof. Let $J_1 \subset J \cap \mathfrak{p}$ be a J_K-class. By Lemma 2.5.8 there exists a J_K-class $J_2 \subset J \cap \mathfrak{p}$ such that J_2 is well-behaved w.r.t. $K.e$, and Corollary 3.3.3 gives $k \in G^\theta$ such that $J_1 = k.J_2$. Since k defines an automorphism of the symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$, the class $J_1 = k.J_2$ is well-behaved w.r.t. $K.(k.e) = k.(K.e)$. □

3.3.2 Case AIII (1)

We fix $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k}, \mathfrak{p}) = (\mathfrak{g}, \mathfrak{k}^{\Phi}, \mathfrak{p}^{\Phi})$ of type AIII as in section 3.1.4 and we use the notation introduced in 3.2.2. For simplicity we assume that the numbers N_a, N_b are such that $N_b \leq N_a$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace. Since the involutions of type AIII are conjugate, and the Cartan subspaces are K-conjugate, one can find a Cartan subalgebra $\mathfrak{h}'$ containing $\mathfrak{a}$ and satisfying the following conditions (see, for example, [GW, Polarizations-Type AIII, p. 20]). There exists a basis $(\varpi_1, \dots, \varpi_N)$ of $\mathfrak{h}'^*$ such that: $\varpi_j(t)$, $1 \leq j \leq N$, are the eigenvalues of $t \in \mathfrak{h}'$ and $B = \{\alpha_j = \varpi_j - \varpi_{j+1} \mid 1 \leq j \leq N-1\}$ is a σ-fundamental system of the root system $R = R(\mathfrak{g}, \mathfrak{h}')$. Recall that the Weyl group $W(\mathfrak{g}, \mathfrak{h}') = N_G(\mathfrak{h}')/Z_G(\mathfrak{h}')$ can be naturally identified with the group $\mathfrak{S}(\{\varpi_1, \dots, \varpi_N\}) \cong \mathfrak{S}_N = \mathfrak{S}([\![1, N]\!])$, where we denote by $\mathfrak{S}(E)$ the permutation group of a set E. Moreover, the action of θ on $\mathfrak{h}'$ is defined by:

$$\varpi_i(\theta(t)) = \begin{cases} \varpi_{N+1-i}(t) & \text{if } \min(i, N+1-i) \leqslant N_b; \\ \varpi_i(t) & \text{otherwise.} \end{cases} \tag{3.7}$$

Fix the semisimple part s of an element belonging to $J \cap \mathfrak{p}$. By Lemma 2.4.2, $J \cap \mathfrak{p}$ is the union of J_K-classes of the form $K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^{\bullet} + n)$ where $n \in \mathfrak{p}^s$ is nilpotent. Thanks to Proposition 2.3.5 we may assume that $s \in \mathfrak{a}_{\mathbb{Q}}$ is in the positive Weyl chamber defined by B. Recall from (3.2) that we write

$$\mathfrak{g}^s = \bigoplus_{c \in \mathsf{sp}(s)} \mathfrak{w}_{s,c}, \quad \mathfrak{w}_{s,c} = \mathfrak{gl}(V_{s,c}),$$

where $\mathfrak{gl}(V_{s,c})$ is naturally embedded in $\mathfrak{g} = \mathfrak{gl}(V)$. Note that $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s) = \bigoplus_{c \in \mathsf{sp}(s)} \Bbbk \, \mathrm{Id}_{V_{s,c}}$. Let $g \in N_G(\mathfrak{g}^s)$; then $s' = g.s \in \mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)$, hence $s'_{|V_{s,c}} = c' \, \mathrm{Id}_{V_{s,c}}$ for some $c' \in \mathsf{sp}(s)$, that is to say $V_{s,c} \subset V_{s',c'}$. It is then easily seen that the map $\eta : c \mapsto c'$ defines a permutation of $\mathsf{sp}(s)$ such that $V_{s,c} = V_{s',c'}$. If $\mathtt{r}(g) = \eta^{-1} \in \mathfrak{S}(\mathsf{sp}(s))$ one has $V_{s',c} = g.V_{s,c} = V_{s,\mathtt{r}(g)(c)}$ and it follows that:

$$g.\mathfrak{w}_{s,c} = \mathfrak{w}_{s,\mathtt{r}(g)(c)} \quad \text{for all } c \in \mathsf{sp}(s).$$

From this observation one deduces a group homomorphism

$$\mathtt{r} : N_G(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)) = N_G(\mathfrak{g}^s) \longrightarrow \mathfrak{S}(\mathsf{sp}(s)), \quad g \mapsto \mathtt{r}(g).$$

Clearly, if $\gamma = \mathtt{r}(g)$ one has:

$$m(s, \gamma(c)) = m(s, c) \quad \text{for all } c \in \mathsf{sp}(s). \tag{3.8}$$

This condition characterizes the elements of the image of $\mathtt{r}$:

Lemma 3.3.6. *An element $\gamma \in \mathfrak{S}(\mathsf{sp}(s))$ is in the image of the morphism $\mathtt{r}$ if and only if it satisfies (3.8).*

Proof. Let $c_1, \dots, c_\ell$ be the distinct eigenvalues of s. By construction, γ can be identified with the element $\gamma \in \mathfrak{S}_\ell$ such that $\gamma(c_i) = c_{\gamma(i)}$, $1 \leq i \leq \ell$. Write $[\![1, N]\!]$ as a disjoint union $\bigsqcup_{j=1}^{\ell} J_j$, where $J_j = \{k : \varpi_k(s) = c_j\}$. By (3.8) one has $\#J_j = m(s, c_j) = \#J_{\gamma(j)} = m(s, \gamma(c_j))$. One can therefore find $w \in \mathfrak{S}_N \cong W(\mathfrak{g}, \mathfrak{h}')$ such that $w(J_j) = J_{\gamma(j)}$ for $j = 1, \dots, \ell$. Let $g \in N_G(\mathfrak{h}')$ be a representative of w. One then gets $g \in N_G(\mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}^s)) = N_G(\mathfrak{g}^s)$ and $\mathtt{r}(g) = \gamma$. □

Recall from §3.2.2 that we denote by $\mathsf{sp}_{\pm}(s)$ the set of classes $\{\pm c : c \in \mathsf{sp}'(s)\}$. For every $k \in N_K(\mathfrak{g}^s)$ we have

$$\mathfrak{w}_{s,\mathtt{r}(k)(-c)} = k.\mathfrak{w}_{s,-c} = k.\theta(\mathfrak{w}_{s,c}) = \theta(k.\mathfrak{w}_{s,c}) = \mathfrak{w}_{s,-\mathtt{r}(k)(c)}.$$

Thus $\mathtt{r}(k)(-c) = -\mathtt{r}(k)(c)$ and, since $\mathfrak{g}_{s,\pm c} = \mathfrak{w}_{s,c} \oplus \mathfrak{w}_{s,-c}$, one gets $k.\mathfrak{g}_{s,\pm c} = \mathfrak{g}_{s,\pm\mathtt{r}(k)(c)}$. Therefore, any element of $\mathtt{r}(N_K(\mathfrak{g}^s))$ induces a permutation of $\mathsf{sp}_{\pm}(s)$. By Lemma 3.2.4, if $0 \in \mathsf{sp}(s)$, the factor $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$ is the unique factor of type AIII in the decomposition of the symmetric Lie algebra $(\mathfrak{g}^s, \mathfrak{k}^s)$ and, as $k \in N_K(\mathfrak{g}^s)$ defines an automorphism of this symmetric pair, one necessarily has $\mathtt{r}(k)(0) = 0$. It follows that $\mathtt{r}$ induces a homomorphism:

$$\mathtt{r}' : N_K(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)) = N_K(\mathfrak{g}^s) \longrightarrow \mathfrak{S}(\mathsf{sp}_{\pm}(s) \smallsetminus \{0\}), \quad k \mapsto \mathtt{r}'(k),$$

with the convention that $\mathsf{sp}_{\pm}(s) \smallsetminus \{0\} = \mathsf{sp}_{\pm}(s)$ when $0 \notin \mathsf{sp}(s)$.

Lemma 3.3.7. (1) *Let $c_0, c_1 \in \mathsf{sp}(s) \smallsetminus \{0\}$ be such that $m(s, c_0) = m(s, c_1)$. There exists $k \in N_K(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s))$ such that: $\mathtt{r}'(k)(\pm c_i) = \mathtt{r}'(k)(\pm c_{1-i})$, for $i = 0, 1$, and $\mathtt{r}'(k)(\pm c) = \pm c$ for all $\pm c \in \mathsf{sp}_{\pm}(s) \smallsetminus \{\pm c_0, \pm c_1\}$.*
(2) *A permutation γ of $\mathsf{sp}_{\pm}(s) \smallsetminus \{0\}$ belongs to $\mathtt{r}'(N_K(\mathfrak{g}^s))$ if and only if*

$$m(s, \pm c) = m(s, \gamma(\pm c)) \ \text{ for all } \pm c \in \mathsf{sp}_{\pm}(s) \smallsetminus \{0\}.$$

In particular, for such a permutation γ there exists $k \in N_K(\mathfrak{g}^s)$ such that

$$k.\mathfrak{g}_{s,\pm c} = \mathfrak{g}_{s,\gamma(\pm c)}$$

where γ is, if necessary, extended to $\mathsf{sp}_{\pm}(s)$ by $\gamma(0) = 0$.

Proof. (1) Recall that $s \in \mathfrak{a}_{\mathbb{Q}}$ is in the positive Weyl chamber defined by B. Therefore, for $i = 0, 1$, $I_i = \{j \mid c_i = \varpi_j(s)\} \subset [\![1, N]\!]$ is an interval; set $I_i = [\![d_i^1, d_i^2]\!]$. In the case $\pm c_0 = \pm c_1$ the element $k = \mathrm{Id}$ obviously works. Otherwise, we may replace c_i by $-c_i$ to ensure that $d_i^2 \leqslant N_b \leqslant N/2$ and we define a permutation $\gamma \in \mathfrak{S}_N$ by:

$$\gamma(j) = \begin{cases} j - d_i^1 + d_{1-i}^1 & \text{if } j \in I_i; \\ j & \text{if } j \leqslant (N+1)/2 \text{ and } j \notin I_1 \cup I_2; \\ N + 1 - \gamma(N + 1 - j) & \text{if } j > (N+1)/2. \end{cases}$$

One has: $\varpi_j(s) = \pm c_{1-i}$ if $\varpi_j(s) = \pm c_i$, $i = 0, 1$ and $\pm\varpi_{\gamma(j)}(s) = \pm\varpi_j(s)$ otherwise. Denote by $w \in W = W(\mathfrak{g}, \mathfrak{h}') \cong \mathfrak{S}_N$ the element corresponding to the permutation γ, hence $w.\varpi_j = \varpi_{\gamma(j)}$. From (3.7) one deduces that:

$$\theta(w.\varpi_j) = \begin{cases} \varpi_{N+1-\gamma(j)} = \varpi_{\gamma(N+1-j)} = w.\theta(\varpi_j) \text{ if } \min(j, N+1-j) \leqslant N_b; \\ \varpi_{\gamma(j)} = w.\theta(\varpi_j) \text{ otherwise.} \end{cases}$$

This implies $\theta \circ w(\alpha) = w \circ \theta(\alpha)$ for all $\alpha \in R(\mathfrak{g}, \mathfrak{h}')$; thus θ commutes with w, i.e. $w \in W_\sigma$ in the notation of §2.2. By Remark 2.2.4(2) there exists $k \in K$ acting like w on $\mathfrak{h}'$. Therefore $k \in N_K(\mathfrak{c}_\mathfrak{g}(\mathfrak{g}^s))$, $\mathtt{r}(k) = \gamma$ and k has the desired properties.
(2) It suffices to write an element of $\mathfrak{S}(\mathsf{sp}_\pm(s) \smallsetminus \{0\})$ as a product of transpositions and to apply part (1). $\square$

If $x = t + n \in \mathfrak{g}^s$ we write $x = \sum_c x_{s,c} = \sum_c (t_{s,c} + n_{s,c})$ where $t_{s,c} + n_{s,c}$ is the Jordan decomposition of $x_{s,c} \in \mathfrak{w}_{s,c}$ (thus $n_{s,c}$ is the nilpotent part of $x_{s,c}$).

We first state consequences of Lemma 3.2.4 for a nilpotent element $x = n \in \mathfrak{p}^s$. As θ sends $n_{s,c}$ onto $-n_{s,-c}$, the Young diagram of $n_{s,c} \in \mathfrak{w}_{s,c}$ is the same as the Young diagram of $n_{s,-c} \in \mathfrak{w}_{s,-c}$. Morevover, the $(K^s)^\circ$-orbit of n in $\mathfrak{p}^s$ is characterized by the Young diagrams of the $n_{s,c}$ for $c \neq 0$ and the ab-diagram of $n_{s,0}$.

Lemma 3.3.8. *Let $x = t + n$ and $x' = t' + n'$ be G-conjugate elements of $\mathfrak{p}$ with $t, t' \in \mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^\bullet$. Then $n_{s,0}$ and $n'_{s,0}$ have the same Young diagram. Furthermore, if x and x' are K-conjugate, $n_{s,0}$ and $n'_{s,0}$ have the same ab-diagram.*

Proof. If $m(s, 0) \leqslant 1$ one has $n_{s,0} = n'_{s,0} = 0$; we will therefore assume that $0 \in \mathsf{sp}(s)$ and $\mathfrak{w}'_{s,0} = \mathfrak{sl}(V_{s,0}) \neq \{0\}$. One can define equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ on $\mathsf{sp}(s)$ as follows. Say that $c\mathcal{R}d$ if the two following conditions are satisfied: $\mathfrak{w}_{s,c}$ is isomorphic to $\mathfrak{w}_{s,d}$, i.e. $m(s, c) = m(s, d)$, and $n_{s,c}, n_{s,d}$ have the same Young diagram. The relation $\mathcal{R}'$ is defined similarly with n' instead of n. As observed above, the elements c and $-c$ are in the same equivalence class. Consequently, the class containing 0 is the only class, for $\mathcal{R}$ or $\mathcal{R}'$, having odd cardinality.
Since $t, t' \in \mathfrak{c}_\mathfrak{g}(\mathfrak{g}^s)^\bullet$, there exists $g \in N_G(\mathfrak{g}^s)$ such that $g.x' = x$ and we can set $\gamma = \mathtt{r}(g)$. One then has $n_{s,\gamma(c)} = g.n'_{s,c}$, therefore γ sends each $\mathcal{R}'$-equivalence class to an $\mathcal{R}$-equivalence class. Thus, as the cardinality of the equivalence class of $\gamma(0)$ is odd, $\gamma(0)\mathcal{R}0$, $g.n'_{s,0} = n_{s,\gamma(0)}$ and $n_{s,0}, n'_{s,0}$ have the same Young diagram. This proves the first statement.
Now assume that $g \in K$, hence $g \in N_K(\mathfrak{g}^s)$. We have already shown before Lemma 3.3.7 that, in this situation, $\gamma(0) = 0$. Thus $g.n_{s,0} = n'_{s,0}$ with $g \in K$, as desired. $\square$

Let $y = t + n \in J \cap \mathfrak{p}$. Then $(\mathfrak{g}_{t,0}, \mathfrak{k}_{t,0})$ is either (0) or a reductive factor of type AIII. By Lemma 3.2.4 the parameter of this factor is the same as the parameter of $(\mathfrak{g}, \mathfrak{k})$, thus it does not depend on the choice of $y \in J \cap \mathfrak{p}$. Recall that $n_{t,0}$ is the component of n lying in $\mathfrak{g}_{t,0} = \mathfrak{w}_{t,0}$ and define $\Gamma^\Phi(y)$ to be the ab-diagram of $n_{t,0}$ in $(\mathfrak{g}_{t,0}, \mathfrak{k}_{t,0})$. Remark that one can recover the ab-diagram of $n_{t,0}$ in $(\mathfrak{g}, \mathfrak{k})$ by adding to $\Gamma^\Phi(y)$ some pairs of rows of length 1, one row beginning by a and the other by b.

Proposition 3.3.9. (1) *Let $x^1, x^2 \in J \cap \mathfrak{p}$. The following conditions are equivalent*

$$\text{(i) } \Gamma^\Phi(x^1) = \Gamma^\Phi(x^2) \;\; ; \qquad \text{(ii) } J_K(x^1) = J_K(x^2).$$

Set $\Gamma^\Phi(J_K(x)) = \Gamma^\Phi(x)$ for $x \in J \cap \mathfrak{p}$.

(2) *The map* $J_1 \mapsto \Gamma^\Phi(J_1)$ *gives an injection from the set of* J_K*-classes contained in* $J \cap \mathfrak{p}$ *to the set of admissible ab-diagrams for the symmetric pair* $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$.

Proof. (1) Write $x^i = t^i + n^i$ for $i = 1, 2$. By Lemma 2.4.2 there exists $k^i \in K$ such that $k^i.t^i \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet$. Observe that $\Gamma^\Phi(k^i.x^i) = \Gamma^\Phi(x^i)$ and $J_K(k^i.x^i) = J_K(x^i)$, therefore we may assume that $x^i \in \mathfrak{g}^s$ and $t^i \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet$ for $i = 1, 2$. We may also assume that $m(t^1, 0) = m(t^2, 0) \geqslant 1$, otherwise each $n^i_{t^i,0} = 0$ is zero and the equivalence is clear.
As $n^i_{t^i,0}$ belongs to the unique simple factor of type AIII of $(\mathfrak{g}^{t^i}, \mathfrak{k}^{t^i})$, one has $n^i_{t^i,0} \in \mathfrak{w}_{s,0}$, thus $n^i_{t^i,0} = n^i_{s,0}$ and we can set $n^i_0 := n^i_{t^i,0} = n^i_{s,0}$. For $0 \neq c \in \mathsf{sp}(s)$, set $n^i_c = n^i_{s,c}$. Recall that the J_K-class of x^i is $J_K(x^i) = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n^i)$.
(ii) $\Rightarrow$ (i): By hypothesis there exists an element of $K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n^1)$ which is K-conjugate to x^2. Lemma 3.3.8 then shows that n^1_0 has the same ab-diagram as n^2_0 for the pair $(\mathfrak{g}, \mathfrak{k})$, which implies that $\Gamma^\Phi(x^1) = \Gamma^\Phi(x^2)$ (cf. remark above).
(i) $\Rightarrow$ (ii): Suppose that n^1_0 and n^2_0 have the same ab-diagram in $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$. We want to show that n^1 is $N_K(\mathfrak{g}^s)$-conjugate to n^2. Observe that n^1_0 and n^2_0 have the same orbit under the group $K_{s,0}$, where we set $K_{s,\pm c} = (G_{s,\pm c} \cap K)^\circ$. As n^1 is G-conjugate to n^2 there exists $g \in N_G(\mathfrak{g}^s)$ such that $g.n^1_c = n^2_{\gamma(c)}$, which defines $\gamma = \mathfrak{r}(g) \in \mathfrak{S}(\mathsf{sp}(s))$. Since n^i_c, n^i_{-c} have the same diagrams for all c, there exists $\gamma' \in \mathfrak{S}(\mathsf{sp}(s))$ such that:

$$\mathfrak{w}_{s,c} \cong \mathfrak{w}_{s,\gamma'(c)}, \qquad n^1_c \text{ has the same diagram as } n^2_{\gamma'(c)}, \qquad \gamma'(-c) = -\gamma'(c),$$

for all $c \in \mathsf{sp}(s)$. The permutation γ' fixes 0 and induces $\gamma'' \in \mathfrak{S}(\mathsf{sp}_\pm(s))$. Lemma 3.3.7(2) gives an element $k \in N_K(\mathfrak{g}^s)$ such that $k.\mathfrak{g}_{s,\pm c} = \mathfrak{g}_{s,\pm\gamma'(c)}$ for $c \in \mathsf{sp}_\pm(s)$. Set $n^3 = k.n^1$; then n^3_c has the same diagram as n^2_c for all $c \neq 0$, and the same ab-diagram when $c = 0$. By the results on type A0, $n^3_c + n^3_{-c}$ and $n^2_c + n^2_{-c}$ are $K_{s,\pm c}$-conjugate for $c \neq 0$. This proves the existence of $k' \in Z_K(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet) \subset N_K(\mathfrak{g}^s)$ such that $k'.n^3 = n^2$ and $k'k.n^1 = n^2$. In particular, $K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n^1) = K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n^2)$ and the result follows.
(2) is an obvious consequence of (1). ☐

Remark. On can show, by a similar proof to that of Proposition 3.3.9, that condition (2.8) of section 2.5 holds in case AIII.

3.3.3 Case AIII (2)

We continue with the same notation. Thus: $e \in \mathfrak{g} = \mathfrak{gl}_N$ is a nilpotent element, the partition of N associated to $\mathcal{O} = G.e$ is denoted by $\boldsymbol{\lambda} = (\lambda_1 \geqslant \cdots \geqslant \lambda_{\delta_\mathcal{O}})$, $\Phi : [\![1, \delta_\mathcal{O}]\!] \longrightarrow \{a, b\}$ is an arbitrary function and $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k}, \mathfrak{p}) = (\mathfrak{g}, \mathfrak{k}^\Phi, \mathfrak{p}^\Phi)$ is the symmetric Lie algebra defined in §3.1.4, hence $e \in \mathfrak{p} = \mathfrak{p}^\Phi$. As above, $S = S_G$ is the G-sheet containing e and J is a J_G-class of S intersecting $\mathfrak{p}$. Recall from section 2.5 that the set $\{g.e\}_{g \in \mathsf{Z}}$ parameterizes the K-orbits $\mathcal{O}_{g.e} = K.(g.e)$ contained in $\mathcal{O} \cap \mathfrak{p}$. We aim to show that the condition (♣) introduced in 2.5 holds, see Proposition 3.3.11.

Let $\Gamma_1 := \Delta(\Phi)$ be the admissible ab-diagram associated to $e \in \mathfrak{p}^\Phi$ and let $J_1 \subset J \cap \mathfrak{p}$ be a J_K-class. By Theorem 3.2.1 and Lemma 1.6.1 the conditions (♡) and (*) are satisfied; therefore, Lemma 2.5.8(iii) can be applied in this situation. Let J_2 be given by this lemma (for $g = \mathrm{Id}$), thus $J_2 \subset J$ is a J_K-class which is well behaved w.r.t $\mathcal{O}_e$. Set $Y := J_2 \cap (e + X_\mathfrak{p}) \subset J \cap (\mathfrak{q} \cap \mathfrak{p})^\bullet$; as observed in Remark 2.5.7, we have:

$$\dim Y = \dim J \cap \mathfrak{p} - m. \tag{3.9}$$

Let s be the semisimple part of an element of $J \cap \mathfrak{p}$ and recall that $\Gamma^\Phi(J_1)$, resp. $\Gamma^\Phi(J_2)$, is the admissible ab-diagram, for $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$, associated to J_1, resp. J_2, by Proposition 3.3.9(2). We are going to compare these diagrams with Γ_1 in order to obtain an element $g.e$ ($g \in \mathsf{Z}$) such that J_1 is well behaved w.r.t $\mathcal{O}_{g.e}$.

Let $\mathfrak{q} = \bigoplus_i \mathfrak{q}_i$ be as in 1.6.1 and $x = s + n$ be an element of $J \cap (\mathfrak{q} \cap \mathfrak{p})^\bullet$, cf. Corollary 3.2.2. Recall that we write $n = \sum_{i=1}^{\delta_\mathcal{O}} n_i$ with $n_i \in \mathfrak{q}_i$. Let $\mathcal{O}' \subset \mathfrak{g}_{s,0}$ be the nilpotent orbit $G_{s,0}.n_{s,0}$ and let $\boldsymbol{\mu} = (\mu_1 \geqslant \cdots \geqslant \mu_{\delta_{\mathcal{O}'}})$ be the associated partition of $m(s,0)$. Remark that the shape of the Young diagram underlying $\Gamma^\Phi(J_1)$ or, equivalently, $\Gamma^\Phi(J_2)$, is given by $\boldsymbol{\mu}$.
On the other hand $n = \sum_{c \in \mathsf{sp}(s)} n_{s,c}$ with $n_{s,c} \in \mathfrak{w}_{s,c}$ and, by Corollary 3.2.9, one can write $n_{s,0} = \sum_i n_{i,s,0}$ where each $n_{i,s,0} \in \mathfrak{q}_{i,s,0} \cap \mathfrak{p}^\Phi$ is regular. This yields in particular that $\delta_{\mathcal{O}'} \leqslant \delta_\mathcal{O}$. We can therefore define a map

$$\flat^\Phi(x) : [\![1, \delta_{\mathcal{O}'}]\!] \longrightarrow \{a, b\} \tag{3.10}$$

where $\flat^\Phi(x)(i)$ is the first symbol of the one row ab-diagram of $n_{i,s,0} \in \mathfrak{q}_{i,s,0} \cap \mathfrak{p}^\Phi$. Observe that when λ_i is odd, Corollary 3.2.9(3-4) yields

$$\mu_i \equiv 1 \bmod 2 \ \text{ and } \flat^\Phi(x) = \Phi(i) \text{ for all } x \in J \cap (\mathfrak{q} \cap \mathfrak{p})^\bullet. \tag{3.11}$$

It is not difficult to see that the ab-diagram $\Delta(\flat^\Phi(x))$ associated to the function $\flat^\Phi(x)$, see §3.1.4, coincides with the ab-diagram $\Gamma^\Phi(x)$ defined before Proposition 3.3.9. Thus, according to the previous notation:

$$\Delta(\flat^\Phi(y)) = \Gamma^\Phi(y) = \Gamma^\Phi(J_2) \ \text{ for all } y \in Y \subset J_2 \cap (\mathfrak{q} \cap \mathfrak{p})^\bullet.$$

Remark. One may have $\flat^\Phi(x) \neq \flat^\Phi(x')$ with $K.x' = K.x$. Such examples can be easily obtained by permuting blocks $\mathfrak{q}_i$ and $\mathfrak{q}_j$ such that $\lambda_i = \lambda_j$.

Now, let $\Psi' : [\![1, \delta_{\mathcal{O}'}]\!] \longrightarrow \{a, b\}$ be a map such that its associated ab-diagram, $\Delta(\Psi')$, is equal to $\Gamma^\Phi(J_1)$. Under this notation, we want to construct $\Psi : [\![1, \delta_\mathcal{O}]\!] \to \{a, b\}$ such that $\Psi' = \flat^\Psi(y)$ and

$$\Delta(\flat^\Psi(y)) = \Gamma^\Psi(y) = \Gamma^\Phi(J_1) \ \text{ for all } y \in Y.$$

Fix $y \in Y$ and define Ψ as follows:

$$\begin{cases} \Psi(i) = \Phi(i) \ \text{ if } \Psi'(i) = \flat^\Phi(y)(i) \text{ and } i \leqslant \delta_{\mathcal{O}'}; \\ \Psi(i) \neq \Phi(i) \ \text{ if } \Psi'(i) \neq \flat^\Phi(y)(i) \text{ and } i \leqslant \delta_{\mathcal{O}'}; \\ \Psi(i) = \Phi(i) \ \text{ for } i \in [\![\delta_{\mathcal{O}'} + 1, \delta_\mathcal{O}]\!]. \end{cases}$$

By (3.11), for each $i \in [\![1, \delta_{\mathcal{O}}]\!]$ such that λ_i is odd one has $\Psi'(i) = \Psi(i)$.

Lemma 3.3.10. *The ab-diagram $\Gamma_2 := \Delta(\Psi)$ is admissible for the symmetric pair $(\mathfrak{g}, \mathfrak{k}^\Phi)$.*

Proof. The only thing to prove is that N'_a (resp. N'_b), the number of a (resp. b) in Γ_2 is equal to N_a (resp. N_b). This is equivalent to showing that $N'_a - N'_b = N_a - N_b$. From (3.11) and the definition of Ψ one deduces:

$$\begin{aligned}
&N_a - N_b - (N'_a - N'_b)\\
&\quad = \#\{i \mid \Phi(i) = a \text{ and } \lambda_i \equiv 1 \bmod 2\} - \#\{i \mid \Phi(i) = b \text{ and } \lambda_i \equiv 1 \bmod 2\}\\
&\qquad - \#\{i \mid \Psi(i) = a \text{ and } \lambda_i \equiv 1 \bmod 2\} + \#\{i \mid \Psi(i) = b \text{ and } \lambda_i \equiv 1 \bmod 2\}\\
&\quad = \#\{i \mid \flat^\Phi(y)(i) = a \text{ and } \lambda_i \equiv 1 \bmod 2\} - \#\{i \mid \flat^\Phi(y)(i) = b \text{ and } \lambda_i \equiv 1 \bmod 2\}\\
&\qquad - \#\{i \mid \Psi'(i) = a \text{ and } \lambda_i \equiv 1 \bmod 2\} + \#\{i \mid \Psi'(i) = a \text{ and } \lambda_i \equiv 1 \bmod 2\} \qquad .
\end{aligned}$$

Since the diagrams $\Delta(\Psi') = \Gamma^\Phi(J_1)$ and $\Delta(\flat^\Phi(y)) = \Gamma^\Phi(J_2)$ are admissible in the same symmetric pair $(\mathfrak{g}_{s,0}, \mathfrak{k}_{s,0})$, the previous equation implies that $N_a - N_b - (N'_a - N'_b) = 0$. □

From the function Ψ one constructs, as in §3.1.4, the symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k}', \mathfrak{p}') = (\mathfrak{g}, \mathfrak{k}^\Psi, \mathfrak{p}^\Psi)$ with $V = V_a^\Psi \bigoplus V_b^\Psi$. Since $\mathfrak{q}_i \cap \mathfrak{k}$ and $\mathfrak{q}_i \cap \mathfrak{k}'$ are both spanned by even sup- and sub-diagonals, we obtain the same symmetric Lie subalgebras $(\mathfrak{q}_i, \mathfrak{q}_i \cap \mathfrak{k}, \mathfrak{q}_i \cap \mathfrak{p}) = (\mathfrak{q}_i, \mathfrak{q}_i \cap \mathfrak{k}', \mathfrak{q}_i \cap \mathfrak{p}')$. It follows that the function $\flat^\Psi(z) : [\![1, \delta_{\mathcal{O}'}]\!] \to \{a, b\}$ is well defined for all $z \in J \cap (\mathfrak{q} \cap \mathfrak{p})^\bullet = J \cap (\mathfrak{q} \cap \mathfrak{p}')^\bullet$.

Recall that $y \in (\mathfrak{q} \cap \mathfrak{p})^\bullet = (\mathfrak{q} \cap \mathfrak{p}')^\bullet$, thus $\flat^\Psi(y)$ is defined; we claim that $\flat^\Psi(y) = \Psi'$. Set $V_a^\Phi(i) = \langle v_j^{(i)} : 1 \leq j \leq \lambda_i \rangle \cap V_a^\Phi$, $V_b^\Phi(i) = \langle v_j^{(i)} : 1 \leq j \leq \lambda_i \rangle \cap V_b^\Phi$, and define $V_a^\Psi(i), V_b^\Psi(i)$ accordingly. Observe that: $V_a^\Phi(i) = V_a^\Psi(i)$, $V_b^\Phi(i) = V_b^\Psi(i)$ when $\Phi(i) = \Psi(i)$, and $V_a^\Phi(i) = V_b^\Psi(i)$, $V_b^\Phi(i) = V_a^\Psi(i)$ otherwise. Suppose that $\Phi(i) \neq \Psi(i)$; by definition of $\flat^\Phi, \flat^\Psi$ one has $\flat^\Phi(y)(i) \neq \flat^\Psi(y)(i)$, therefore $\flat^\Psi(y)(i) = \Psi'(i)$ by definition of Ψ. The equality $\flat^\Psi(y)(i) = \Psi'(i)$ is obtained in the same way when $\Phi(i) = \Psi(i)$. The equality $\flat^\Psi(y) = \Psi'$ implies in particular $\Gamma^\Psi(y) = \Gamma^\Phi(J_1)$.

We can now show that the condition (♣) is satified in type AIII:

Proposition 3.3.11. *For each J_K-class $J_1 \subset J \cap \mathfrak{p}$, there exists $g \in \mathsf{Z}$ such that J_1 is well-behaved w.r.t. $\mathcal{O}_{g.e}$*

Proof. By Lemma 3.3.10 one can find $g' \in \mathrm{GL}_N$ such that $g'.V_a^\Psi = V_a^\Phi$ and $g'.V_b^\Psi = V_b^\Phi$. Then, $g = \rho(g') \in G$ induces an isomorphism of symmetric Lie algebras between $(\mathfrak{g}, \mathfrak{k}', \mathfrak{p}')$ and $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ (cf. end of 3.1.4). Since $e \in \mathfrak{q} \cap \mathfrak{p}$, one has $e \in \mathfrak{p}'$ and $g.e \in \mathfrak{p}$; therefore, up to conjugation by an element of K^Φ (the algebraic group associated to $\mathfrak{k} = \mathfrak{k}^\Phi$), we may assume that $g \in \mathsf{Z}$ (see §3.1.5). These remarks imply that $\Gamma^\Phi(g.y) = \Gamma^\Psi(y) = \Gamma^\Phi(J_1)$ is the ab-diagram associated to J_1 with respect to Φ, cf. Proposition 3.3.9. From $Y \subset \mathfrak{q} \cap \mathfrak{p} = \mathfrak{q} \cap \mathfrak{p}'$ one gets $g.y \in g.Y \subset J \cap \mathfrak{p}$ and, since $g.Y$ is irreducible, one has $g.Y \subset J_1$. In particular, $g.Y \subset g.(e + X_{\mathfrak{p}}(g.\mathscr{S})) \cap \mathfrak{p} \subset g.e + X_{\mathfrak{p}}(g.\mathscr{S})$ is contained in J_1 with $\dim g.Y = \dim J_1 - m$. The result then follows from Remarks 2.5.5 and 2.5.7. □

4 Main theorem and remarks

4.1 Main theorem

In this subsection we give the description of the K-sheets when $(\mathfrak{g}, \theta)$ is of type A. Thus, $\mathfrak{g} \cong \mathfrak{gl}_N$ and $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ is a symmetric Lie algebra. Suppose that $S_G \subset \mathfrak{g}$ is a G-sheet intersecting $\mathfrak{p}$. In (2.6), cf. Remark 2.5.9, we have defined, for any nilpotent element $e \in S_G \cap \mathfrak{p}$ and any normal $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, h, f)$, the following subvariety of $S_G \cap \mathfrak{p}$:

$$S_K(S_G, \mathscr{S}) = S_K(\mathscr{S}) = S_K(K.e) = \overline{K.(e + X_{\mathfrak{p}}(\mathscr{S}))}^{\bullet}.$$

We aim to describe the K-sheets and the varieties $S_G \cap \mathfrak{p}$ in terms of the $S_K(K.e)$.

Recall from Remark 2.5.4(2) that $S_G \cap \mathfrak{p}$ is smooth; in particular, its irreducible components are disjoint. The next lemma reduces the study of K-sheets to the study of irreducible components of $S_G \cap \mathfrak{p}$; this result may be false in some cases of type 0, see the remark previous to Corollary 2.1.3.

Lemma 4.1.1. *Let S_G be a G-sheet of $\mathfrak{g}$ intersecting $\mathfrak{p}$, then each irreducible component of $S_G \cap \mathfrak{p}$ is a K-sheet.*

Proof. Let S_K be an irreducible component of $S_G \cap \mathfrak{p}$. As $S_G \cap \mathfrak{p}$ is a union of K-orbits of same dimension, there exists a K-sheet S'_K containing S_K. Recall that, as $\mathfrak{g} \cong \mathfrak{gl}_N$, two distinct G-sheets are disjoint (see the discussion previous to Corollary 2.1.3). It follows that S'_K must be contained in S_G and, therefore, in $S_G \cap \mathfrak{p}$. This proves that $S'_K = S_K$, hence the result. □

Theorem 4.1.2. (i) *The K-sheets of $\mathfrak{p}$ are disjoint, they are exactly the smooth irreducible varieties $S_K(\mathcal{O}_K)$ where $\mathcal{O}_K \subset \mathfrak{p}$ is a nilpotent K-orbit.*
(ii) *Let S_G be a G-sheet intersecting $\mathfrak{p}$. Then, $S_G \cap \mathfrak{p}$ is a smooth equidimensional variety and each of its irreducible component is some $S_K(\mathcal{O}_K)$, where $\mathcal{O}_K \subset S_G \cap \mathfrak{p}$ is a nilpotent K-orbit.*
(iii) *Let $S_K \subset \mathfrak{p}$ be a K-sheet and e be a nilpotent element of S_K embedded in a normal $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, h, f)$. Define Y by $e + Y = S_K \cap (e + \mathfrak{p}^f)$. Then $S_K = \overline{K.(e + Y)}^{\bullet}$.*

Proof. We need to summarize the conditions introduced in §2.5 and proved in cases AI, AII and AIII: (♡) has been proved in Theorem 3.2.1 (with proof in Proposition 3.2.6 for type AIII); (◇) was established in Remark 3.2.3 (types AI, AII) and Remark 3.2.8 (type AIII); (♣) has been obtained in Corollary 3.3.5 (types AI, AII) and Proposition 3.3.11 (type AIII).
Claim (ii) is therefore consequence of Remark 2.5.4(2) (or equivalently Proposition 2.5.3) and Theorem 2.5.11.
Recall that G-sheets are disjoint. Then, from $\mathfrak{p}^{(m)} \subset \mathfrak{g}^{(2m)}$, it follows that each K-sheet is contained in a unique G-sheet. So, (i) is consequence of (ii) and Lemma 4.1.1.
Under the hypothesis in (iii), e belongs to S_K, hence $S_K = S_K(\mathscr{S})$ is the unique K-sheet containing e. Therefore,

$$e + Y \subset e + X(\mathscr{S}) \cap \mathfrak{p} \subset S_K(\mathscr{S}) \cap (e + \mathfrak{p}^f) = e + Y.$$

The assertion in (iii) then follows from the definition of $S_K(\mathscr{S})$. □

Remark. One can be more precise about the number of irreducible components of $S_G \cap \mathfrak{p}$, see §4.2(4).

Fix a sheet S_G intersecting $\mathfrak{p}$. One can compute the dimension of $S_G \cap \mathfrak{p}$ in terms of the partitions associated to the nilpotent orbit $\mathcal{O} \subset S_G$. Let $\boldsymbol{\lambda} = (\lambda_1 \geqslant \cdots \geqslant \lambda_{\delta_{\mathcal{O}}})$ and $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}_1 \geqslant \cdots \geqslant \tilde{\lambda}_{\delta_{\mathfrak{t}}})$ be the partitions of N defined in 1.6.1. Pick $e \in \mathcal{O} \cap \mathfrak{p}$ and recall that if $\mathscr{S} = (e,h,f)$ is a normal $\mathfrak{sl}_2$-triple we set $S_K(K.e) = \overline{K.(e + X_{\mathfrak{p}}(\mathscr{S}))}^{\bullet}$.

Proposition 4.1.3. *Under the previous notation one has*

$$\dim S_G \cap \mathfrak{p} = \dim S_K(K.e) = \lambda_1 + \frac{1}{2}\Big(N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2\Big)$$

in types AI *and* AII, *and*

$$\dim S_G \cap \mathfrak{p} = \dim S_K(K.e) = \sum_{i=1}^{\delta_{\mathcal{O}}} \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor + \frac{1}{2}\Big(N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2\Big).$$

in type AIII.

Proof. Recall that $\dim G.e = N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2$, see [CM], and $\dim K.e = \frac{1}{2}\dim G.e$. By Theorem 4.1.2 and Remark 2.5.5 one has

$$\dim S_G \cap \mathfrak{p} = \dim S_K(K.e) = \dim K.e + \dim X_{\mathfrak{p}}(\mathscr{S}).$$

We know that $X_{\mathfrak{p}}(\mathscr{S}) = X(\mathscr{S})$ in types AI and AII, cf. Theorem 3.2.1. Therefore, Remark 1.4.8 and equation (1.8) yield $\dim S_G \cap \mathfrak{p} = \dim K.e + \dim X(\mathscr{S}) = \dim K.e + \dim \mathfrak{t} = \dim K.e + \lambda_1$. Hence:

$$\dim S_G \cap \mathfrak{p} = \lambda_1 + \frac{1}{2}\Big(N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2\Big).$$

Since the morphism ε is quasi-finite, see Remark 1.4.8, one has $\dim X_{\mathfrak{p}}(\mathscr{S}) = \dim \mathfrak{c}$ in type AIII by Proposition 3.2.7. It then follows from (3.6) that

$$\dim S_G \cap \mathfrak{p} = \dim K.e + \dim \mathfrak{c} = \dim K.e + \sum_{i=1}^{\delta_{\mathcal{O}}} \left\lfloor \tfrac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor.$$

Thus

$$\dim S_G \cap \mathfrak{p} = \sum_{i=1}^{\delta_{\mathcal{O}}} \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor + \frac{1}{2}\Big(N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2\Big)$$

as desired. □

4.2 Remarks and comments

We collect here various remarks and comments about the results obtained in the previous sections. To keep the length of the exposition reasonable we will not give full details of the proofs, leaving them to the interested reader.
If not otherwise specified, we assume that $(\mathfrak{g}, \theta) \cong (\mathfrak{gl}_N, \theta)$ is of type AI-II-III; we then retain the notation of Section 3 and §4.1. In particular, $S_G \subset \mathfrak{g}$ is a G-sheet which intersects $\mathfrak{p}$, $\mathcal{O} = G.e$, $e \in S_G \cap \mathfrak{p}$, is the nilpotent orbit contained in S_G, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{\delta_{\mathcal{O}}})$ is the associated partition of N, $\mathbf{v}$ is the basis of V introduced in §1.6.1, $e + X = e + X(\mathscr{S})$, with $\mathscr{S} = (e, h, f)$, is a Slodowy slice of S_G, $X_{\mathfrak{p}} = X_{\mathfrak{p}}(\mathscr{S}) = X \cap \mathfrak{p}$, $\mathfrak{c} \subset \mathfrak{t}$ is such that $\varepsilon(e + \mathfrak{c}) = e + X_{\mathfrak{p}}$ in case AIII (cf. (3.5)), etc.
For simplicity, we will sometimes assume that $\mathfrak{g} = \mathfrak{sl}_N$. When this is the case, the above notation refers to their intersection with $\mathfrak{sl}_N$.

(1) Theorems 3.2.1 and 4.1.2 show that $e + X_{\mathfrak{p}}$ is "almost" a slice for $S_G \cap \mathfrak{p}$, or for a K-sheet contained in S_G and containing e, meaning that the G-orbit of any element of $S_G \cap \mathfrak{p}$ intersects $e + X_{\mathfrak{p}}$. But, contrary to the Lie algebra case, $e + X_{\mathfrak{p}}$ does not necessarily intersect each K-orbit contained in the given K-sheet, even when (2.8) is satisfied. This phenomenon already occurs for the regular sheet in $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}_2, \mathfrak{so}_2)$. Indeed, using the one parameter subgroup $(F_t)_{t \in \Bbbk^\times}$ introduced in §1.4, it is possible to show that $K.(e + X_{\mathfrak{p}})$ contains only one nilpotent orbit, namely $K.e$, while $S_G^{reg} \cap \mathfrak{p} = \mathfrak{p}^{reg}$ is the regular K-sheet of $\mathfrak{p}$ and contains two nilpotent K-orbits.
Nevertheless, by [KR, Theorem 11] $e + X_{\mathfrak{p}} = e + \mathfrak{p}^f$ is a set of representative elements for G^θ-orbits in $\mathfrak{p}^{reg}$. Using Proposition 3.3.2 and Theorem 3.2.1, one can show that $G^\theta.(e + X_{\mathfrak{p}}) = S_G \cap \mathfrak{p}$ in type AI and AII. In type AIII this result is far of being true, as shown by the following example.

Example. Consider a symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k})$ isomorphic to $(\mathfrak{sl}_4, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \Bbbk)$. Let S_G be the G-sheet containing the nilpotent orbit associated to the partition $\boldsymbol{\lambda} = (2, 2)$. Choose the nilpotent element $e \in S_G$ in Jordan canonical form and fix a function Φ determining the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. If we set $\Phi(1) = \Phi(2) = a$, the ab-diagram of e given by these choices is $\Gamma^\Phi(e) = \Delta(\Phi) = \substack{ab \\ ab}$, see 3.1.4. Here, $\mathfrak{t} = \mathfrak{c}$ is the set of diagonal elements of the form $\mathrm{diag}(c, -c, c, -c)$, $c \in \Bbbk$ (cf. (1.7) and (3.5)). Then, $S_G = G.(e + \mathfrak{t})$ contains exactly two Jordan G-classes: $J_G^{nil} = G.e$ (obtained for $c = 0$) and $J_G^{ss} = G.\mathfrak{t}$. Moreover, one has $e + X = \varepsilon(e + \mathfrak{t}) \subset \mathfrak{p}$, see Proposition 3.2.7.
If $x \in J_G^{ss} \cap \mathfrak{p}$, the slice $e + X$ contains a unique element y such that $G.x = G.y$ (cf. Lemma 1.6.1) and, by Proposition 2.2.6, y is K-conjugate to x. Again, it is possible to show that e is the unique nilpotent element in $e + X$. Therefore, $J_K^{ss} = J_G^{ss} \cap \mathfrak{p} = K.((e + X) \smallsetminus \{e\})$ is the dense J_K-class contained in $S_K(K.e)$. The same results hold for each $g.e$ such that $g \in \mathsf{Z}$, see 3.1.5, and one gets that $S_G \cap \mathfrak{p} = \overline{J_K^{ss}}^{\bullet} = S_K(K.e)$.
A study of the centralizers of nilpotent elements then shows that $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k}).(e + X)$ does not contain the K-orbit of the nilpotent element with ab-diagram $\substack{ab \\ ba}$. This proves that $e + X \cap \mathfrak{p}$ is

not a slice of $S_K(K.e) = S_G \cap \mathfrak{p}$ for the action of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$. $\square$

(2) Suppose that $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{k})$ is an arbitrary reductive symmetric Lie algebra. Recall [TY, 39.4] that a G-sheet containing a semisimple element is called a *Dixmier sheet*. Similarly, we will say that a K-sheet which contains a semisimple element is a *Dixmier K-sheet*.
If $\mathfrak{g}$ is semisimple of type A, all G-sheets are Dixmier sheets, cf. [Kr, 2.3]. This implies that, for each sheet S_G and $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, h, f)$ as in §1.4, the set $e + X(S_G, \mathscr{S}) = e + X(\mathscr{S})$ contains a semisimple element. For symmetric pairs of type AI or AII, the K-sheets are all of the form $S_K(\mathscr{S}) = S_K(K.e) = \overline{K.(e + X(\mathscr{S}))}^{\bullet}$ (cf. Theorems 3.2.1 and 4.1.2); thus, in these cases, any K-sheet is a Dixmier K-sheet.
In type AIII there exist K-sheets containing no semisimple element and one can characterize them in terms of the partition $\boldsymbol{\lambda}$ associated to the nilpotent element $e \in S_G \cap \mathfrak{p}$. Recall first that we have shown (see Propositions 3.2.7 and 3.2.6) that

$$G.(S_G \cap \mathfrak{p}) = G.(e + X_{\mathfrak{p}}) = G.(e + \mathfrak{c}).$$

Therefore, if $e+\mathfrak{c}$ contains a semisimple element, the same is true for $e+X_{\mathfrak{p}}(\mathscr{S})$ and $S_K(K.e) = \overline{K.(e + X_{\mathfrak{p}})}^{\bullet}$. Conversely, if a K-sheet in $S_G \cap \mathfrak{p}$ is Dixmier, then $e + \mathfrak{c}$ contains a semisimple element. Thus, a K-sheet contained in $S_G \cap \mathfrak{p}$ is Dixmier if and only if $e+\mathfrak{c}$ contains a semisimple element. By Corollary 1.6.4, an element $e + c$, $c = \sum_i c_i \in \mathfrak{c} \subset \mathfrak{h} = \bigoplus_i \mathfrak{h}_i$, is semisimple if and only if the eigenvalues of each c_i are distinct. It follows from the construction of $\mathfrak{c} = \alpha(\mathfrak{c}')$, see (3.4), that this condition reduces to: 0 is an eigenvalue of c_i of multiplicity at most one for all i. One then deduces from the definition of $\mathfrak{c}'$ the following result:

Claim 4.2.1. *In type* AIII*, a K-sheet is Dixmier if and only if the partition $\boldsymbol{\lambda}$ satisfies: $\lambda_i - \lambda_{i+1}$ is odd for at most one $i \in [\![1, \delta_{\mathcal{O}}]\!]$ (where we set $\lambda_{\delta_{\mathcal{O}}+1} = 0$).* $\square$

Observe that the condition for a K-sheet to be Dixmier depends only on the nilpotent orbit $G.e$ and that $S_K(K.e)$ is Dixmier if and only if $S_K(K.g.e)$, $g \in \mathsf{Z}$, is Dixmier.

(3) Recall from Section 2.5 that a nilpotent orbit of $\mathfrak{g}$ is *rigid* when it is a sheet of $[\mathfrak{g}, \mathfrak{g}]$. When $\mathfrak{g}$ is of type A the only rigid nilpotent orbit is $\{0\}$. In other cases it may happen that a rigid orbit $\mathcal{O}_1$ contains a non-rigid orbit $\mathcal{O}_2$ in its closure (see the classification of rigid nilpotent orbits in [CM]). Observe that, since the nilpotent cone is closed, a sheet containing $\mathcal{O}_2$ cannot be contained in the closure of $\mathcal{O}_1$. One gets in this way some sheets whose closure is not a union of sheets. One can ask if similar facts occur for symmetric pairs $(\mathfrak{g}, \mathfrak{k})$, in particular when $\mathfrak{g}$ is of type A.
Let $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ be a symmetric Lie algebra; a nilpotent K-orbit in $\mathfrak{p}$ which is a K-sheet in $\mathfrak{p} \cap [\mathfrak{g}, \mathfrak{g}]$ will be called rigid. We remarked in (2) that, in types AI and AII, each K-sheet contains a semisimple element; thus, $\{0\}$ is the only rigid nilpotent K-orbit in these cases. Assume that $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ is of type AIII, $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{k}$, and recall from the proof of Proposition 4.1.3 (using Remark 1.4.8) that $S_K(K.e) = K.e$ if and only if $\dim \mathfrak{c} = 0$. The arguments given in (2) about K-sheets can be adapted to prove:

Claim 4.2.2. *The orbit $K.e$ is rigid if and only if the partition $\boldsymbol{\lambda}$ satisfies: $\lambda_i - \lambda_{i+1} \leqslant 1$ for all $i \in [\![1, \delta_{\mathcal{O}}]\!]$.* □

Note that the previous result depends only on the partition $\boldsymbol{\lambda}$ and not on the ab-diagram of e. In particular, $K.e$ is rigid if and only if each K-orbit contained in $G.e \cap \mathfrak{p}$ is rigid.

Example. Consider the symmetric pair $(\mathfrak{gl}_6, \mathfrak{gl}_3 \oplus \mathfrak{gl}_3)$ and a rigid K-orbit $\mathcal{O}_1$ associated to the partition $\boldsymbol{\lambda} = (3, 2, 1)$. This orbit contains in its closure a nilpotent K-orbit $\mathcal{O}_2$ with partition $(3, 1, 1, 1)$, cf. [Oh2], but $\mathcal{O}_2$ is not rigid.
In type AIII, we can construct in this way K-sheets whose closures are not a union of sheets. □

(4) We have shown in Theorem 4.1.2 that the irreducible components of $S_G \cap \mathfrak{p}$ are K-sheets and are of the form $S_K(\mathcal{O}_K)$, where $\mathcal{O}_K$ is a (nilpotent) K-orbit contained in $\mathcal{O} = G.e$. The number of these irreducible components thus depends on the analysis of the equality $S_K(\mathcal{O}_K^1) = S_K(\mathcal{O}_K^2)$ where $\mathcal{O}_K^1, \mathcal{O}_K^2$ are nilpotent K-orbits. An obvious necessary condition is $\mathcal{O} = G.\mathcal{O}_K^1 = G.\mathcal{O}_K^2$. We begin by showing that $S_G \cap \mathfrak{p}$ is irreducible in types AI and AII.
Assume that $(\mathfrak{g}, \mathfrak{k})$ is of type AII. Then, Proposition 3.3.2 and the equality $K = G^\theta$ imply that $\mathcal{O} \cap \mathfrak{p} = \mathcal{O}_K^1$. Therefore, the map $\mathcal{O}_K \mapsto S_K(\mathcal{O}_K)$ is in this case bijective and $S_G \cap \mathfrak{p}$ is a single K-sheet.
Assume that $(\mathfrak{g}, \mathfrak{k})$ is of type AI and choose $\omega \in K^\theta \smallsetminus K$ (if it exists). We need to compare $S_K(\mathcal{O}_K^1)$ and $S_K(\mathcal{O}_K^2)$ when $\mathcal{O}_K^1 = \omega.\mathcal{O}_K^2$. Since $\omega \in N_G(\mathfrak{p})$, one has $S_K(\mathcal{O}_K^1) = \omega.S_K(\mathcal{O}_K^2)$. As the K-sheets are Dixmier, cf. (2), $S_K(\mathcal{O}_K^1)$ contains a J_K-class J_K^{ss} of semisimple elements. But Lemma 2.2.6 then implies that J_K^{ss} is stable under ω; thus, $S_K(\mathcal{O}_K^1) = S_K(\mathcal{O}_K^2)$ since distinct K-sheets are disjoint. Hence, in type AI, $S_G \cap \mathfrak{p}$ is a K-sheet.
The situation in type AIII is more complicated and one can find G-sheets having a nonirreducible intersection with $\mathfrak{p}$. The characterization of the equality $S_K(\mathcal{O}_K^1) = S_K(\mathcal{O}_K^2)$ is given in Claim 4.2.3. We will simply sketch the proof of this result, which can be decomposed in two steps. First, one has to characterize the unique J_G-class J such that $S_G \cap \mathfrak{p} = \overline{J \cap \mathfrak{p}}^{\bullet}$, see Theorem 2.5.11. By the same theorem we know that $S_K(\mathcal{O}_K^i) = \overline{J_i}^{\bullet}$ for a unique J_K-class J_i. Since Proposition 3.3.9 says that J_i is determined by its ab-diagram, it remains to relate this ab-diagam to that of $\mathcal{O}_K^i$.
In order to state Claim 4.2.3 we first have to define the notion of "rigidified ab-diagram". Let Γ be an ab-diagram coresponding to a nilpotent K-orbit $\mathcal{O}_K \subset \mathfrak{p}$; remove from Γ the maximum number of pairs of consecutive columns of the same length. The new ab-diagram obtained in this way is uniquely determined and is called the the *rigidified* ab-diagram deduced from Γ, or associated to $\mathcal{O}_K$. The terminology can be justified by the following remark: a rigidified ab-diagram corresponds to a rigid nilpotent K-orbit in some other symmetric pair of type AIII.

Claim 4.2.3. *The two orbits $\mathcal{O}_K^1$ and $\mathcal{O}_K^2$ are contained in the same K-sheet, i.e. $S_K(\mathcal{O}_K^1) = S_K(\mathcal{O}_K^2)$, if and only if their associated rigidified ab-diagrams are equal.* □

Example. Let $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}_8, \mathfrak{gl}_4 \oplus \mathfrak{gl}_4)$ and $\mathcal{O}$ be the nilpotent G-orbit with associated partition $\boldsymbol{\lambda} = (4, 3, 1)$. The set $\mathcal{O} \cap \mathfrak{p}$ splits into four K-orbits $\mathcal{O}_K^j$, $1 \leq j \leq 4$, whose respective ab-diagrams are

$$\Gamma(\mathcal{O}_K^1) = \begin{array}{l} abab \\ aba \\ b \end{array} \;;\; \Gamma(\mathcal{O}_K^2) = \begin{array}{l} abab \\ bab \\ a \end{array} \;;\; \Gamma(\mathcal{O}_K^3) = \begin{array}{l} baba \\ aba \\ b \end{array} \;;\; \Gamma(\mathcal{O}_K^4) = \begin{array}{l} baba \\ bab \\ a \end{array} .$$

The associated rigidified ab-diagrams are, respectively:

$$\begin{array}{l} ab \\ a \\ b \end{array} \;; \qquad \begin{array}{l} ab \\ a \\ b \end{array} \;; \qquad \begin{array}{l} ba \\ a \\ b \end{array} \;; \qquad \begin{array}{l} ba \\ a \\ b \end{array} .$$

The previous result implies that $S_G \cap \mathfrak{p}$ is the disjoint union of $S_K(\mathcal{O}_K^1) = S_K(\mathcal{O}_K^2)$ and $S_K(\mathcal{O}_K^3) = S_K(\mathcal{O}_K^4)$. $\square$

The proof of Claim 4.2.3 being rather technical, we will only try to give below an idea of the main ingredients. Define first a family $(\ell_i)_{i \in [\![1,\delta_{\mathcal{O}}]\!]}$ of integers by:

$$\ell_{\delta_{\mathcal{O}}} := \left\lfloor \frac{\lambda_{\delta_{\mathcal{O}}}}{2} \right\rfloor, \quad \ell_i := \ell_{i+1} + \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor.$$

Note that $\ell := \sum_i \ell_i = \dim \mathfrak{c}$ (see the construction of $\mathfrak{c}'$ in (3.4)). Then, for any ℓ-tuple $(t_1, \dots, t_\ell) \in \Bbbk^\ell$, define $y(t_1, \dots t_\ell) \in \mathfrak{q} \subset \mathfrak{g}$ by its action on $\mathbf{v}$:

$$y(t_1, \dots, t_\ell).v_j^{(i)} = \begin{cases} t_1 v_2^{(i)} & \text{if } j = 1; \\ (1 + t_{j/2}) v_{j-1}^{(i)} & \text{if } j \in 2[\![1, \ell_i]\!]; \\ t_{(j+1)/2} v_{j+1}^{(i)} + v_{j-1}^{(i)} & \text{if } j \in 2[\![1, \ell_i - 1]\!] + 1; \\ v_{j-1}^{(i)} & \text{otherwise.} \end{cases}$$

Set $Y = \{y(t_1, \dots, t_\ell) \mid (t_1, \dots, t_\ell) \in (\Bbbk \setminus \{-1\})^\ell\}$. Is is easily seen that $e \in Y$ and one can check that

$$G.Y = G.(e + \mathfrak{c}) = G.(S_G \cap \mathfrak{p}), \quad Y \subset S_G \cap (\mathfrak{q} \cap \mathfrak{p})^\bullet, \tag{4.1}$$

see Proposition 3.2.7 for the second equality. Recall that J_K-classes are locally closed and observe that Y is irreducible; it follows that there exists a unique J_K-class J_1 such that $\overline{Y \cap J_1} = \overline{Y}$. Set $Y_1 = Y \cap J_1$ and let $J \subset S_G$ be the J_G-class containing J_1. From (4.1) we deduce:

$$S_G \cap \mathfrak{p} \subseteq (G.\overline{Y_1}^\bullet) \cap \mathfrak{p} \subseteq (G.\overline{J_1}^\bullet) \cap \mathfrak{p} \subseteq \overline{J \cap \mathfrak{p}}^\bullet \subseteq S_G \cap \mathfrak{p}.$$

Hence, we can get in this way the Jordan G-class J such that $S_G \cap \mathfrak{p} = \overline{J \cap \mathfrak{p}}^\bullet$. The datum $(\mathfrak{l}, \mathcal{O}')$ of J can be computed from any element of Y_1. One finds in particular that the Young diagram of $\mathcal{O}'$ can be obtained from the Young diagram of $\mathcal{O}$ by removing pairs of consecutive columns of the same length. Now, as prevoulsy said, we need to compare the ab-diagram of $\mathcal{O}_K^i$ with the ab-diagram of the dense J_K-class containded in $S_K(\mathcal{O}_K^i)$. We have observed after Claim 4.2.2 that the rigidity property depends only on $\mathcal{O}$, thus, we may assume that $\mathcal{O}_K^i = K.e$. Recall that the K-sheets are disjoint, cf. Theorem 4.1.2; as $e \in \overline{J_1}$ this forces J_1 to be the dense J_K-class in $S_K(\mathcal{O}_K^i)$. Since $Y_1 \subset J \cap (\mathfrak{q} \cap \mathfrak{p})^\bullet$, the ab-diagram $\Gamma^\Phi(J_1)$ associated to J_1, cf. Proposition 3.3.9, can be computed with the help of the function $\flat^\Phi$ defined in (3.10). One finds that $\flat^\Phi(x)(i) = \Phi(i)$ for all $x \in Y_1$ and $i \in [\![1, \delta_{\mathcal{O}}]\!]$. It is then straightforward to check that the rigidified ab-diagram associated to $K.e$ is equal to $\Gamma^\Phi(J_1)$.

(5) If $(\mathfrak{g}, \mathfrak{k}, \mathfrak{p})$ is a semisimple symmetric Lie algebra, one may try to use the description of K-sheets to study the irreducible components of the commuting variety $\mathfrak{C}(\mathfrak{p}) := \{(x, y) \in \mathfrak{p} \times \mathfrak{p} \mid [x, y] = 0\}$. Assume that $x = s+n \in \mathfrak{p}$ (Jordan decomposition) and $y \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n$. Then, $\mathfrak{g}^x = \mathfrak{g}^y$ and $(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n, \mathfrak{p}^x) \subset \mathfrak{C}(\mathfrak{p})$ is irreducible; consequently, $\mathfrak{C}(x) := \overline{K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n, \mathfrak{p}^x)} \subset \mathfrak{C}(\mathfrak{p})$ (where K acts diagonally on $\mathfrak{p} \times \mathfrak{p}$) is also irreducible. As $\mathfrak{C}(x) = \mathfrak{C}(y)$ if $J_K(x) = J_K(y)$, we can define $\mathfrak{C}(J_K(x)) := \mathfrak{C}(x)$. A computation using Grassmannians shows that $(y, \mathfrak{p}^y) \subset \mathfrak{C}(x)$ when $y \in \overline{J_K(x)}^\bullet$. If S is a K-sheet and J is a dense Jordan K-class in S, one can therefore set $\mathfrak{C}(S) := \mathfrak{C}(J)$. Observe that $\mathfrak{C}(S)$ contains all the pairs $(x, y) \in \mathfrak{C}(\mathfrak{p})$ with $x \in S$. It follows that $\mathfrak{C}(\mathfrak{p})$ decomposes in a finite union of irreducible varieties $\bigcup_S \mathfrak{C}(S)$ where S runs over K-sheets. Using the fact that $\mathrm{GL}_2 .J_K(x)$ is an irreducible subvariety of $\mathfrak{C}(\mathfrak{p})$, where the group GL_2 acts naturally on $\mathfrak{C}(\mathfrak{p}) \subset \mathfrak{p} \times \mathfrak{p}$ (see, for example [Bu1]), one can show that if $(x, y) \in \mathfrak{C}(\mathfrak{p})$ with $y \notin \overline{J_K(x)}$, then $\mathfrak{C}(J_K(x))$ is not an irreducible component of $\mathfrak{C}(\mathfrak{p})$. Recall that $z \in \mathfrak{p}$ is said to be $\mathfrak{p}$-distinguished if 0 is the only semisimple element in $\mathfrak{p}^z$ [TY, 38.10.1].

Claim 4.2.4. *Let $x = s + n$ be such that $\mathfrak{C}(J_K(x))$ is an irreducible component of $\mathfrak{C}(\mathfrak{p})$. Then n is $\mathfrak{p}$-distinguished in $[\mathfrak{g}^s, \mathfrak{g}^s] \cap \mathfrak{p}$.*

Proof. Suppose on the contrary that there exists a semisimple element $0 \neq s' \in [\mathfrak{g}^s, \mathfrak{g}^s] \cap \mathfrak{p}$ such that $[s', n] = 0$ ans set $x = s + s'$. It follows easily from the description of the closures of J_G-classes given in [TY, 39.2.2] that the semisimple part of an element of $\overline{J_G(x)}$ is conjugate to an element of $\overline{J_G(s)}$. Since x' is semisimple and $\dim \mathfrak{g}^{x'} < \dim \mathfrak{g}^x$ we have $x' \notin \overline{J_G(s)}$, hence $x' \notin \overline{J_G(x)}$. Therefore, $(x', x) \in \mathrm{GL}_2 .\mathfrak{C}(J_K(x)) \smallsetminus \mathfrak{C}(J_K(x))$, giving a contradiction. □

Recall that the K-orbits of $\mathfrak{p}$-distinguished elements are classified, see [PT] or [Bu1]. Using Claim 4.2.4, in order to find an upper bound to the number of irreducible components of $\mathfrak{C}(\mathfrak{p})$ one can look for K-sheets containing a dense Jordan class of the form $K.(\mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)^\bullet + n)$, where the element n is $\mathfrak{p}$-distinguished in $[\mathfrak{g}^s, \mathfrak{g}^s]$.

In type AI or AII, each sheet is Dixmier, cf. (2), and therefore contains a dense J_K-class consisting of semisimple elements [TY, 39.6.7]. Under the previous notation, this forces the $\mathfrak{p}$-distinguished element n to be zero, which says that $\mathfrak{a} = \mathfrak{c}_{\mathfrak{p}}(\mathfrak{p}^s)$ is a Cartan subspace. Hence, $\mathfrak{C}(\mathfrak{p}) = \overline{K.(\mathfrak{a} \times \mathfrak{a})}$ is irreducible and we recover a result proved in [Pa1, Pa2].

In type AIII we are going to illustrate the method in symmetric rank 1 and 2. Recall that the symmetric rank is the dimension of any Cartan subspace, so in this case it is equal to $\min(N_a, N_b)$. In the symmetric rank one case with $N_a > N_b = 1$ (see §3.1.4), following [Pa1] one can obtain in this way that $\mathfrak{C}(\mathfrak{p})$ has three irreducible components. In the symmetric rank two case we will see what happens when $N_b = 2$ and $N_a \geqslant 4$, the other cases being quite similar. For simplicity, we remove the center and assume $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}_6, \mathfrak{sl}_4 \oplus \mathfrak{sl}_2 \oplus \Bbbk)$. We find that $\mathfrak{C}(\mathfrak{p})$

has at most seven irreducible components, while five is given as a lower bound in [PY]. To be more explicit, it is shown in [PY] that $\mathfrak{C}(\mathfrak{p}) = \bigcup_{q=0}^{4} P_q$ where the P_q are distinct closed subsets such that $P_q \not\subset \bigcup_{q' \neq q} P_{q'}$ for each q. Applying our method, one gets

$$P_2 = \overline{K.(\mathfrak{a} \times \mathfrak{a})}, \quad P_1 = \mathfrak{C}(J_1) \cup \mathfrak{C}(\mathcal{O}_K^1), \quad P_0 = \mathfrak{C}(\mathcal{O}_K^0),$$

where $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{p}$, J_1 is a non-nilpotent J_K-class and $\mathcal{O}_K^1, \mathcal{O}_K^0$ are nilpotent K-orbits. Furthermore, neither $\mathfrak{C}(J_1)$ nor $\mathfrak{C}(\mathcal{O}_K^1)$ is contained in $\bigcup_{q \neq 1} P_q$. The description of the variety P_3, resp. P_4, is analogous to that of P_1, resp. P_0. Since all the elements of $\mathfrak{C}(\mathcal{O}_K^1)$ are nilpotent, the determination of the number of irreducible components of $\mathfrak{C}(\mathfrak{p})$ reduces in this case to the question: do we have $\mathfrak{C}(\mathcal{O}_K^1) \subset \mathfrak{C}(J_1)$?
Unfortunately, the upper bound we obtain increases rapidly with the symmetric rank of $(\mathfrak{g}, \mathfrak{k})$ and we cannot give precise results on the number of irreducible components of $\mathfrak{C}(\mathfrak{p})$.

(6) A natural problem is, using section §2.5, to generalize the results obtained in type A to other types. The action of ε is well described in [IH] for classical Lie algebras and one may ask if conditions ($\heartsuit$), ($\diamondsuit$) or ($\clubsuit$) hold in this case. Concerning ($\heartsuit$), the author made some calculations when $(\mathfrak{g}, \mathfrak{k})$ is of type CI. Im-Hof, cf. [IH], splits this type in three cases that we label CI-I, CI-II and CI-III. It is likely that ($\heartsuit$) remains true for the first two cases. In case CI-III one finds the following counterexample. Consider $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sp}_6, \mathfrak{gl}_3)$ and the sheet S_G with datum $(\mathfrak{l}, 0)$ where $\mathfrak{l}$ is isomorphic to $\mathfrak{gl}_2 \oplus \mathfrak{sp}_2$. Let e and e' be nilpotent elements in $S_G \cap \mathfrak{p}$ with respective ab-diagrams $\Gamma(e) = {}^{abab}_{ab}$ and $\Gamma(e') = {}^{abab}_{ba}$. Embed e, resp. e', in an $\mathfrak{sl}_2$-triple $\mathscr{S}$, resp. $\mathscr{S}'$. One can show that $\dim X_{\mathfrak{p}}(S_G, \mathscr{S}) = 1$, $\dim X_{\mathfrak{p}}(S_G, \mathscr{S}') = 2$ and we then get $G.(e + X_{\mathfrak{p}}(S_G, \mathscr{S})) \subsetneq G.(e' + X_{\mathfrak{p}}(S_G, \mathscr{S}'))$, showing that ($\heartsuit$) is not satisfied. Moreover, we see that the similarity observed in the case $\mathfrak{g} = \mathfrak{gl}_N$ between properties of $X_{\mathfrak{p}}(S_G, g.\mathscr{S})$ and $X_{\mathfrak{p}}(S_G, \mathscr{S})$, when $g \in \mathsf{Z}$, is no longer valid.

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